

# NOTE ON MMAT 5010: LINEAR ANALYSIS (2017 1ST TERM)

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## 1. LECTURE 1: NORMED SPACES

Throughout this note, we always denote  $\mathbb{K}$  by the real field  $\mathbb{R}$  or the complex field  $\mathbb{C}$ . Let  $\mathbb{N}$  be the set of all natural numbers. Also, we write a sequence of numbers as a function  $x : \{1, 2, \dots\} \rightarrow \mathbb{K}$ .

**Definition 1.1.** Let  $X$  be a vector space over the field  $\mathbb{K}$ . A function  $\|\cdot\| : X \rightarrow \mathbb{R}$  is called a norm on  $X$  if it satisfies the following conditions.

- (i)  $\|x\| \geq 0$  for all  $x \in X$  and  $\|x\| = 0$  if and only if  $x = 0$ .
- (ii)  $\|\alpha x\| = |\alpha| \|x\|$  for all  $\alpha \in \mathbb{K}$  and  $x \in X$ .
- (iii)  $\|x + y\| \leq \|x\| + \|y\|$  for all  $x, y \in X$ .

In this case, the pair  $(X, \|\cdot\|)$  is called a normed space.

Also, the distance between the elements  $x$  and  $y$  in  $X$  is defined by  $\|x - y\|$ .

The following examples are important classes in the study of functional analysis.

**Example 1.2.** Consider  $X = \mathbb{K}^n$ . Put

$$\|x\|_p := \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \quad \text{and} \quad \|x\|_\infty := \max_{i=1, \dots, n} |x_i|$$

for  $1 \leq p < \infty$  and  $x = (x_1, \dots, x_n) \in \mathbb{K}^n$ .

Then  $\|\cdot\|_p$  (called the usual norm as  $p=2$ ) and  $\|\cdot\|_\infty$  (called the sup-norm) all are norms on  $\mathbb{K}^n$ .

**Example 1.3.** Put

$$c_0 := \{(x(i)) : x(i) \in \mathbb{K}, \lim |x(i)| = 0\} \text{ (called the null sequence space)}$$

and

$$\ell^\infty := \{(x(i)) : x(i) \in \mathbb{K}, \sup_i |x(i)| < \infty\}.$$

Then  $c_0$  is a subspace of  $\ell^\infty$ . The sup-norm  $\|\cdot\|_\infty$  on  $\ell^\infty$  is defined by

$$\|x\|_\infty := \sup_i |x(i)|$$

for  $x \in \ell^\infty$ . Let

$$c_{00} := \{(x(i)) : \text{there are only finitly many } x(i) \text{'s are non-zero}\}.$$

Also,  $c_{00}$  is endowed with the sup-norm defined above and is called the finite sequence space.

**Example 1.4.** For  $1 \leq p < \infty$ , put

$$\ell^p := \{(x(i)) : x(i) \in \mathbb{K}, \sum_{i=1}^{\infty} |x(i)|^p < \infty\}.$$

Also,  $\ell^p$  is equipped with the norm

$$\|x\|_p := \left( \sum_{i=1}^{\infty} |x(i)|^p \right)^{\frac{1}{p}}$$

for  $x \in \ell^p$ . Then  $\|\cdot\|_p$  is a norm on  $\ell^p$  (see [2, Section 9.1]).

**Example 1.5.** Let  $C^b(\mathbb{R})$  be the space of all bounded continuous  $\mathbb{R}$ -valued functions  $f$  on  $\mathbb{R}$ . Now  $C^b(\mathbb{R})$  is endowed with the sup-norm, that is,

$$\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|$$

for every  $f \in C^b(\mathbb{R})$ . Then  $\|\cdot\|_\infty$  is a norm on  $C^b(\mathbb{R})$ .

Also, we consider the following subspaces of  $C^b(X)$ .

Let  $C_0(\mathbb{R})$  (resp.  $C_c(\mathbb{R})$ ) be the space of all continuous  $\mathbb{R}$ -valued functions  $f$  on  $\mathbb{R}$  which vanish at infinity (resp. have compact supports), that is, for every  $\varepsilon > 0$ , there is a  $K > 0$  such that  $|f(x)| < \varepsilon$  (resp.  $f(x) \equiv 0$ ) for all  $|x| > K$ .

It is clear that we have  $C_c(\mathbb{R}) \subseteq C_0(\mathbb{R}) \subseteq C^b(\mathbb{R})$ .

Now  $C_0(\mathbb{R})$  and  $C_c(\mathbb{R})$  are endowed with the sup-norm  $\|\cdot\|_\infty$ .

**Notation 1.6.** From now on,  $(X, \|\cdot\|)$  always denotes a normed space over a field  $\mathbb{K}$ .

For  $r > 0$  and  $x \in X$ , let

- (i)  $B(x, r) := \{y \in X : \|x - y\| < r\}$  (called an open ball with the center at  $x$  of radius  $r$ ) and  $B^*(x, r) := \{y \in X : 0 < \|x - y\| < r\}$
- (ii)  $B(x, r) := \{y \in X : \|x - y\| \leq r\}$  (called a closed ball with the center at  $x$  of radius  $r$ ).

Put  $B_X := \{x \in X : \|x\| \leq 1\}$  and  $S_X := \{x \in X : \|x\| = 1\}$  the closed unit ball and the unit sphere of  $X$  respectively.

**Definition 1.7.** Let  $A$  be a subset of  $X$ .

- (i) A point  $a \in A$  is called an interior point of  $A$  if there is  $r > 0$  such that  $B(a, r) \subseteq A$ . Write  $\text{int}(A)$  for the set of all interior points of  $A$ .
- (ii)  $A$  is called an open subset of  $X$  if  $\text{int}(A) = A$ .

**Example 1.8.** We keep the notation as above.

- (i) Let  $\mathbb{Z}$  and  $\mathbb{Q}$  denote the set of all integers and rational numbers respectively. If  $\mathbb{Z}$  and  $\mathbb{Q}$  both are viewed as the subsets of  $\mathbb{R}$ , then  $\text{int}(\mathbb{Z})$  and  $\text{int}(\mathbb{Q})$  both are empty.
- (ii) The open interval  $(0, 1)$  is an open subset of  $\mathbb{R}$  but it is not an open subset of  $\mathbb{R}^2$ . In fact,  $\text{int}(0, 1) = (0, 1)$  if  $(0, 1)$  is considered as a subset of  $\mathbb{R}$  but  $\text{int}(0, 1) = \emptyset$  while  $(0, 1)$  is viewed as a subset of  $\mathbb{R}^2$ .
- (iii) Every open ball is an open subset of  $X$  (**Check!!**).

**Definition 1.9.** We say that a sequence  $(x_n)$  in  $X$  converges to an element  $a \in X$  if  $\lim \|x_n - a\| = 0$ , that is, for any  $\varepsilon > 0$ , there is  $N \in \mathbb{N}$  such that  $\|x_n - a\| < \varepsilon$  for all  $n \geq N$ .

In this case,  $(x_n)$  is said to be convergent and  $a$  is called a limit of the sequence  $(x_n)$ .

**Remark 1.10.**

- (i) If  $(x_n)$  is a convergence sequence in  $X$ , then its limit is unique. In fact, if  $a$  and  $b$  both are the limits of  $(x_n)$ , then we have  $\|a - b\| \leq \|a - x_n\| + \|x_n - b\| \rightarrow 0$ . So,  $\|a - b\| = 0$  which implies that  $a = b$ .

From now on, we write  $\lim x_n$  for the limit of  $(x_n)$  provided the limit exists.

- (ii) The definition of a convergent sequence  $(x_n)$  depends on the underlying space where the sequence  $(x_n)$  sits in. For example, for each  $n = 1, 2, \dots$ , let  $x_n(i) := 1/i$  as  $1 \leq i \leq n$  and  $x_n(i) = 0$  as  $i > n$ . Then  $(x_n)$  is a convergent sequence in  $\ell^\infty$  but it is not convergent in  $c_{00}$ .

**Definition 1.11.** Let  $A$  be a subset of  $X$ .

(i) A point  $z \in X$  is called a *limit point* of  $A$  if for any  $\varepsilon > 0$ , there is an element  $a \in A$  such that  $0 < \|z - a\| < \varepsilon$ , that is,  $B^*(z, \varepsilon) \cap A \neq \emptyset$  for all  $\varepsilon > 0$ .

Furthermore, if  $A$  contains the set of all its limit points, then  $A$  is said to be *closed* in  $X$ .

(ii) The closure of  $A$ , write  $\overline{A}$ , is defined by

$$\overline{A} := A \cup \{z \in X : z \text{ is a limit point of } A\}.$$

**Remark 1.12.** With the notation as above, it is clear that a point  $z \in \overline{A}$  if and only if  $B(z, r) \cap A \neq \emptyset$  for all  $r > 0$ . This is also equivalent to saying that there is a sequence  $(x_n)$  in  $A$  such that  $x_n \rightarrow z$ . In fact, this can be shown by considering  $r = \frac{1}{n}$  for  $n = 1, 2, \dots$

**Proposition 1.13.** With the notation as before, we have the following assertions.

(i)  $A$  is closed in  $X$  if and only if its complement  $X \setminus A$  is open in  $X$ .

(ii) The closure  $\overline{A}$  is the smallest closed subset of  $X$  containing  $A$ . The "smallest" in here means that if  $F$  is a closed subset containing  $A$ , then  $\overline{A} \subseteq F$ .

Consequently,  $A$  is closed if and only if  $\overline{A} = A$ .

*Proof.* If  $A$  is empty, then the assertions (i) and (ii) both are obvious. Now assume that  $A \neq \emptyset$ .

For part (i), let  $C = X \setminus A$  and  $b \in C$ . Suppose that  $A$  is closed in  $X$ . If there exists an element  $b \in C \setminus \text{int}(C)$ , then  $B(b, r) \not\subseteq C$  for all  $r > 0$ . This implies that  $B(b, r) \cap A \neq \emptyset$  for all  $r > 0$  and hence,  $b$  is a limit point of  $A$  since  $b \notin A$ . It contradicts to the closeness of  $A$ . So,  $C = \text{int}(C)$  and thus,  $C$  is open.

For the converse of (i), assume that  $C$  is open in  $X$ . Assume that  $A$  has a limit point  $z$  but  $z \notin A$ . Since  $z \notin A$ ,  $z \in C = \text{int}(C)$  because  $C$  is open. Hence, we can find  $r > 0$  such that  $B(z, r) \subseteq C$ . This gives  $B(z, r) \cap A = \emptyset$ . This contradicts to the assumption of  $z$  being a limit point of  $A$ . So,  $A$  must contain all of its limit points and hence, it is closed.

For part (ii), we first claim that  $\overline{A}$  is closed. Let  $z$  be a limit point of  $\overline{A}$ . Let  $r > 0$ . Then there is  $w \in B^*(z, r) \cap \overline{A}$ . Choose  $0 < r_1 < r$  small enough such that  $B(w, r_1) \subseteq B^*(z, r)$ . Since  $w$  is a limit point of  $A$ , we have  $\emptyset \neq B^*(w, r_1) \cap A \subseteq B^*(z, r) \cap A$ . So,  $z$  is a limit point of  $A$ . Thus,  $z \in \overline{A}$  as required. This implies that  $\overline{A}$  is closed.

It is clear that  $\overline{A}$  is the smallest closed set containing  $A$ .

The last assertion follows from the minimality of the closed sets containing  $A$  immediately.

The proof is finished.  $\square$

**Example 1.14.** Retains all notation as above. We have  $\overline{c_{00}} = c_0 \subseteq \ell^\infty$ .

Consequently,  $c_0$  is a closed subspace of  $\ell^\infty$  but  $c_{00}$  is not.

*Proof.* We first claim that  $\overline{c_{00}} \subseteq c_0$ . Let  $z \in \ell^\infty$ . It suffices to show that if  $z \in \overline{c_{00}}$ , then  $z \in c_0$ , that is,  $\lim_{i \rightarrow \infty} z(i) = 0$ . Let  $\varepsilon > 0$ . Then there is  $x \in B(z, \varepsilon) \cap c_{00}$  and hence, we have  $|x(i) - z(i)| < \varepsilon$  for all  $i = 1, 2, \dots$ . Since  $x \in c_{00}$ , there is  $i_0 \in \mathbb{N}$  such that  $x(i) = 0$  for all  $i \geq i_0$ . Therefore, we have  $|z(i)| = |z(i) - x(i)| < \varepsilon$  for all  $i \geq i_0$ . So,  $z \in c_0$  as desired.

For the reverse inclusion, let  $w \in c_0$ . It needs to show that  $B(w, r) \cap c_{00} \neq \emptyset$  for all  $r > 0$ . Let  $r > 0$ . Since  $w \in c_0$ , there is  $i_0$  such that  $|w(i)| < r$  for all  $i \geq i_0$ . If we let  $x(i) = w(i)$  for  $1 \leq i < i_0$  and  $x(i) = 0$  for  $i \geq i_0$ , then  $x \in c_{00}$  and  $\|x - w\|_\infty := \sup_{i=1,2,\dots} |x(i) - w(i)| < r$  as required.  $\square$

## 2. LECTURE 2: BANACH SPACES

A sequence  $(x_n)$  in  $X$  is called a **Cauchy sequence** if for any  $\varepsilon > 0$ , there is  $N \in \mathbb{N}$  such that  $\|x_m - x_n\| < \varepsilon$  for all  $m, n \geq N$ . We have the following simple observation.

**Lemma 2.1.** *Every convergent sequence in  $X$  is a Cauchy sequence.*

The following notation plays an important role in mathematics.

**Definition 2.2.** *A subset  $A$  of  $X$  is said to be complete if every Cauchy sequence in  $A$  is convergent.*

$X$  is called a **Banach space** if  $X$  is a complete normed space.

**Example 2.3.** *With the notation as above, we have the following examples of Banach spaces.*

- (i) *If  $\mathbb{K}^n$  is equipped with the usual norm, then  $\mathbb{K}^n$  is a Banach space.*
- (ii)  *$\ell^\infty$  is a Banach space. In fact, if  $(x_n)$  is a Cauchy sequence in  $\ell^\infty$ , then for any  $\varepsilon > 0$ , there is  $N \in \mathbb{N}$ , we have*

$$|x_n(i) - x_m(i)| \leq \|x_n - x_m\|_\infty < \varepsilon$$

*for all  $m, n \geq N$  and  $i = 1, 2, \dots$ . Thus, if we fix  $i = 1, 2, \dots$ , then  $(x_n(i))_{n=1}^\infty$  is a Cauchy sequence in  $\mathbb{K}$ . Since  $\mathbb{K}$  is complete, the limit  $\lim_n x_n(i)$  exists in  $\mathbb{K}$  for all  $i = 1, 2, \dots$ . Nor for each  $i = 1, 2, \dots$ , we put  $z(i) := \lim_n x_n(i) \in \mathbb{K}$ . Then we have  $z \in \ell^\infty$  and  $\|z - x_n\|_\infty \rightarrow 0$ . So,  $\lim_n x_n = z \in \ell^\infty$  (**Check !!!!**). Thus  $\ell^\infty$  is a Banach space.*

- (iii)  *$\ell^p$  is a Banach space for  $1 \leq p < \infty$ . The proof is similar to the case of  $\ell^\infty$ .*
- (iv)  *$C[a, b]$  is a Banach space.*
- (v) *Let  $C_0(\mathbb{R})$  be the space of all continuous  $\mathbb{R}$ -valued functions  $f$  on  $\mathbb{R}$  which are vanish at infinity, that is, for every  $\varepsilon > 0$ , there is a  $M > 0$  such that  $|f(x)| < \varepsilon$  for all  $|x| > M$ . Now  $C_0(\mathbb{R})$  is endowed with the sup-norm, that is,*

$$\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|$$

*for every  $f \in C_0(\mathbb{R})$ . Then  $C_0(\mathbb{R})$  is a Banach space.*

**Proposition 2.4.** *Let  $Y$  be a subspace of a Banach space  $X$ . Then  $Y$  is a Banach space if and only if  $Y$  is closed in  $X$ .*

*Proof.* For the necessary condition, we assume that  $Y$  is a Banach space. Let  $z \in \bar{Y}$ . Then there is a convergent sequence  $(y_n)$  in  $Y$  such that  $y_n \rightarrow z$ . Since  $(y_n)$  is convergent, it is also a Cauchy sequence in  $Y$ . Then  $(y_n)$  is also a convergent sequence in  $Y$  because  $Y$  is a Banach space. So,  $z \in Y$ . This implies that  $\bar{Y} = Y$  and hence,  $Y$  is closed.

For the converse statement, assume that  $Y$  is closed. Let  $(z_n)$  be a Cauchy sequence in  $Y$ . Then it is also a Cauchy sequence in  $X$ . Since  $X$  is complete,  $z := \lim z_n$  exists in  $X$ . Note that  $z \in Y$  because  $Y$  is closed. So,  $(z_n)$  is convergent in  $Y$ . Thus,  $Y$  is complete as desired.  $\square$

**Corollary 2.5.**  *$c_0$  is a Banach space but the finite sequence  $c_{00}$  is not.*

**Proposition 2.6.** *Let  $(X, \|\cdot\|)$  be a normed space. Then there is a normed space  $(X_0, \|\cdot\|_0)$ , together with a linear map  $i : X \rightarrow X_0$ , satisfy the following condition.*

- (i)  *$X_0$  is a Banach space.*
- (ii) *The map  $i$  is an isometry, that is,  $\|i(x)\|_0 = \|x\|$  for all  $x \in X$ .*
- (iii) *the image  $i(X)$  is dense in  $X_0$ , that is,  $\overline{i(X)} = X_0$ .*

*Moreover, such pair  $(X_0, i)$  is unique up to isometric isomorphism in the following sense: if  $(W, \|\cdot\|_1)$  is a Banach space and an isometry  $j : X \rightarrow W$  is an isometry such that  $j(\overline{X}) = W$ , then there is an isometric isomorphism  $\psi$  from  $X_0$  onto  $W$  such that*

$$j = \psi \circ i : X \rightarrow X_0 \rightarrow W.$$

*In this case, the pair  $(X_0, i)$  is called the completion of  $X$ .*

**Example 2.7.** Proposition 2.6 cannot give an explicit form of the completion of a given normed space. The following examples are basically due to the uniqueness of the completion.

- (i) If  $X$  is a Banach space, then the completion of  $X$  is itself.
- (ii) By Corollary 2.5, the completion of the finite sequence space  $c_{00}$  is the null sequence space  $c_0$ .
- (iii) The completion of  $C_c(\mathbb{R})$  is  $C_0(\mathbb{R})$ .

**Definition 2.8.** A subset  $A$  of a normed space  $X$  is said to be nowhere dense in  $X$  if  $\text{int}(\overline{A}) = \emptyset$ .

**Example 2.9.**

- (i) The set of all integers  $\mathbb{Z}$  is a nowhere dense subset of  $\mathbb{R}$ .
- (ii) The set  $(0, 1)$  is a nowhere dense subset of  $\mathbb{R}^2$  but it is not a nowhere dense subset of  $\mathbb{R}$ .
- (iii) Let  $A := \{x \in c_{00} : x(n) \geq 0, \text{ for all } n = 1, 2, \dots\}$ . Notice that  $A$  is a closed subset of  $c_{00}$ . We claim that  $\text{int}(A) = \emptyset$ . In fact, let  $a \in A$  and  $r > 0$ . Since  $a \in c_{00}$ , there is  $N$  such that  $a(n) = 0$  for all  $n \geq N$ . Now define  $z \in c_{00}$  by  $z(n) = x(n)$  for  $n \neq N$  and  $z(N) := \frac{-r}{2}$ . Then  $z \in c_{00} \setminus A$  and  $\|z - a\|_\infty < r$ . So,  $\text{int}(A) = \emptyset$  and thus,  $A$  is a nowhere dense subset of  $c_{00}$ .

**Lemma 2.10.** Let  $X$  be a Banach space. We have the following assertions.

- (i) A subset  $A$  of  $X$  is nowhere dense in  $X$  if and only if the complement of  $\overline{A}$  is an open dense subset of  $X$ .
- (ii) If  $(W_n)$  is a sequence of open dense subsets of  $X$ , then  $\bigcap_{n=1}^\infty W_n \neq \emptyset$ .

*Proof.* For (i), let  $z \in X$  and  $r > 0$ . It is clear that we have  $B(z, r) \not\subseteq \overline{A}$  if and only if  $B(z, r) \cap \overline{A}^c \neq \emptyset$ . For (ii), we first fix an element  $x_1 \in W_1$ . Since  $W_1$  is open, then there is  $r_1 > 0$  such that  $B(x_1, r_1) \subseteq W_1$ . Notice that since  $W_2$  is open dense in  $X$ , we can find an element  $x_2 \in B(x_1, r_1) \cap W_2$  and  $0 < r_2 < r_1/2$  such that  $\overline{B(x_2, r_2)} \subseteq B(x_1, r_1) \cap W_2$ . To repeat the same step, we can get a sequence of element  $(x_n)$  in  $X$  and a sequence of positive numbers  $(r_n)$  such that

- (a)  $r_{k+1} < r_k/2$ , and
  - (b)  $\overline{B(x_{k+1}, r_{k+1})} \subseteq B(x_k, r_k) \cap W_{k+1}$
- for all  $k = 1, 2, \dots$

From this, we see that  $(x_k)$  is a Cauchy sequence in  $X$ . Then by the completeness of  $X$ ,  $\lim x_k = a$  exists in  $X$ . It remains to show that  $a \in \bigcap W_k$ . Fix  $N$ . Note that by the condition (b) above, we see that  $x_k \in \overline{B(x_N, r_N)} \subseteq B(x_{N-1}, r_{N-1}) \cap W_N$  for all  $k > N$ . Since  $\overline{B(x_N, r_N)}$  is closed, we see that  $a = \lim x_k \in \overline{B(x_N, r_N)}$ . This implies that  $a \in W_N$ . Therefore,  $\bigcap W_k$  is non-empty as required.  $\square$

**Theorem 2.11. Baire Category Theorem:** Let  $X$  be a Banach space. Suppose that  $X = \bigcup_{n=1}^\infty A_n$  for a sequence of subsets  $(A_n)$  of  $X$ . Then there is  $A_{n_0}$  not nowhere dense in  $X$ .

*Proof.* Suppose that each  $A_n$  is nowhere dense in  $X$ . If we put  $W_n := \overline{A_n}^c$ , then each  $W_n$  is an open dense subset of  $X$  by Lemma 2.10 (i). Lemma 2.10 (ii) implies that  $\bigcap W_n \neq \emptyset$ . This gives

$$X \supseteq \left(\bigcap W_n\right)^c = \bigcup W_n^c = \bigcup \overline{A_n} \supseteq \bigcup A_n = X.$$

This leads to a contradiction. The proof is finished.  $\square$

### 3. LECTURE 3: SERIES IN NORMED SPACES

Throughout this section, let  $X$  be a normed space.

Let  $(x_n)$  be a sequence elements in  $X$ . Now for each  $n = 1, 2, \dots$ , put  $s_n = x_1 + \dots + x_n$  and call the  $n$ -th *partial sum* of a formal series  $\sum_{n=1}^\infty x_n$ .

**Definition 3.1.** With the notation as above, we say that a series  $\sum_{n=1}^{\infty} x_n$  is convergent in  $X$  if the sequence of the sequence of partial sums  $(s_n)$  is convergent in  $X$ . In this case, we also write

$$\sum_{n=1}^{\infty} x_n := \lim_n s_n \in X.$$

Moreover, we say that a series  $\sum_{n=1}^{\infty} x_n$  is absolutely convergent in  $X$  if  $\sum_{n=1}^{\infty} \|x_n\| < \infty$ .

**Lemma 3.2.** Let  $(x_n)$  be a Cauchy sequence in a normed space  $X$ . If  $(x_n)$  has a convergent subsequence in  $X$ , then  $(x_n)$  itself is convergent too.

*Proof.* Let  $(x_{n_k})$  be a convergent subsequence of  $(x_n)$  and let  $L := \lim_k x_{n_k} \in X$ . We are going to show that  $\lim_n x_n = L$ .

Let  $\varepsilon > 0$ . Since  $(x_n)$  is a Cauchy sequence, there is  $N \in \mathbb{N}$  such that  $\|x_m - x_n\| < \varepsilon$  for all  $m, n \geq N$ . On the other hand, since  $\lim_k x_{n_k} = L$ , there is  $K \in \mathbb{N}$  such that  $n_K \geq N$  and  $\|L - x_{n_K}\| < \varepsilon$ . Thus, if  $n \geq n_K$ , we see that  $\|x_n - L\| \leq \|x_n - x_{n_K}\| + \|x_{n_K} - L\| < 2\varepsilon$ . The proof is finished.  $\square$

**Proposition 3.3.** Let  $X$  be a normed space. Then the following statements are equivalent.

- (i)  $X$  is a Banach space.
- (ii) Every absolutely convergent series in  $X$  is convergent.

*Proof.* For showing (i)  $\Rightarrow$  (ii), assume that  $X$  is a Banach space and let  $\sum x_k$  be an absolutely convergent series in  $X$ . Put  $s_n := \sum_{k=1}^n x_k$  the  $n$ -th partial sum of  $\sum x_k$ . Let  $\varepsilon > 0$ . Since the series  $\sum_k x_k$  is absolutely convergent, there is  $N \in \mathbb{N}$  such that  $\sum_{n+1 \leq k \leq n+p} \|x_k\| < \varepsilon$  for all  $n \geq N$

and  $p = 1, 2, \dots$ . This gives  $\|s_{n+p} - s_n\| \leq \sum_{n+1 \leq k \leq n+p} \|x_k\| < \varepsilon$  for all  $n \geq N$  and  $p = 1, 2, \dots$ . Thus,

$(s_n)$  is a Cauchy sequence in  $X$ . Then by the completeness of  $X$ , we see that the series  $\sum x_k$  is convergent in  $X$  as desired.

Now suppose that the condition (ii) holds. Let  $(x_n)$  be a Cauchy sequence in  $X$ . Notice that by the definition of a Cauchy sequence, we can find a subsequence  $(x_{n_k})$  of  $(x_n)$  such that  $\|x_{n_{k+1}} - x_{n_k}\| < 1/2^k$  for all  $k = 1, 2, \dots$ . From this, we see that the series  $\sum_k (x_{n_{k+1}} - x_{n_k})$  is absolutely convergent in  $X$ . Then the condition (ii) tells us that the series  $\sum_k (x_{n_{k+1}} - x_{n_k})$  is convergent in  $X$ . Notice that

$x_{n_m} = x_{n_1} + \sum_{k=1}^m (x_{n_{k+1}} - x_{n_k})$  for all  $m = 1, 2, \dots$ . Therefore,  $(x_{n_k})_{k=1}^{\infty}$  is a convergent subsequence of  $(x_n)$ . Then by Lemma 3.2, we see that  $(x_n)$  is convergent in  $X$ . The proof is finished.  $\square$

Recall that a *basis* of a vector space  $V$  over  $\mathbb{K}$  is a collection of vectors in  $V$ , say  $(v_i)_{i \in I}$ , such that for each element  $x \in V$ , we have a unique expression

$$x = \sum_{i \in I} \alpha_i v_i$$

for some  $\alpha_i \in \mathbb{K}$  and **all**  $\alpha_i = 0$  **except finitely many**.

One of fundamental properties of a vector space is that **every vector space must have a basis**. The proof of this assertion is due to the *Zorn's lemma*.

**Definition 3.4.** A sequence  $(x_n)$  is called a Schauder basis for a normed space  $X$  if for each element  $x \in X$ , there is a unique sequence  $(\alpha_n)$  in  $\mathbb{K}$  such that

$$(3.1) \quad x = \sum_{n=1}^{\infty} \alpha_n x_n.$$

**Remark 3.5.**

- (i) Notice that a Schauder basis must be linearly independent vectors. So, it is clear that every Schauder basis is a vector basis for a finite dimensional vector space. However, a Schauder basis need not be a vector basis for a normed space in general. For example, if we consider the sequence  $(e_n)$  in  $c_0$  given by  $e_n(n) = 1$ ; otherwise,  $e_n(i) = 0$ , then  $(e_n)$  is a Schauder basis for  $c_0$  but it is not a vector basis.
- (ii) In the Definition 3.4, the expression 3.1 depends on the order of  $(x_n)$ . More precise, if  $\sigma : \{1, 2, \dots\} \rightarrow \{1, 2, \dots\}$  is a bijection, then the Eq 3.1 CANNOT assure that we still have the expression  $x = \sum_{n=1}^{\infty} \alpha_{\sigma(n)} x_{\sigma(n)}$  for each  $x \in X$ .

**Example 3.6.** (i) If  $X$  is of finite dimension, then the vector bases are the same as the Schauder bases.

- (ii) Let  $e_n$  be a sequence defined as in Remark 3.5(i), then the sequence  $(e_n)$  is a Schauder basis for the spaces  $c_0$  and  $\ell^p$  for  $1 \leq p < \infty$ .

**Definition 3.7.** A normed space  $X$  is said to be separable if there is a countable dense subset of  $X$ .

**Example 3.8.** (i) The space  $\mathbb{C}^n$  is separable. In fact, it is clear that  $(\mathbb{Q} + i\mathbb{Q})^n$  is a countable dense subset of  $\mathbb{C}^n$ .

- (ii) The space  $\ell^\infty$  is an important example of nonseparable Banach space. In fact, if we put  $D := \{x \in \ell^\infty : x(i) = 0 \text{ or } 1\}$ , then  $D$  is an uncountable subset of  $\ell^\infty$ . Moreover, we have  $\|x - y\|_\infty = 1$  for any  $x, y \in D$  with  $x \neq y$ . Thus,  $\{B(x, 1/2) : x \in D\}$  is an uncountable family of disjoint open balls of  $\ell^\infty$ . So, if  $C$  is a countable dense subset of  $\ell^\infty$ , then  $C \cap B(x, 1/2) \neq \emptyset$  for all  $x \in D$ . Also, for each element  $z \in C$ , there is a unique element  $x \in D$  such that  $z \in B(x, 1/2)$ . It leads to a contradiction since  $D$  is uncountable. Therefore,  $\ell^\infty$  is nonseparable.

**Proposition 3.9.** Let  $X$  be a normed space. Then  $X$  is separable if and only if there is a countable subset  $A$  of  $X$  such that the linear span of  $A$  is dense in  $X$ , that is, for any element  $x \in X$  and  $\varepsilon > 0$ , there are finite many elements  $x_1, \dots, x_N$  in  $A$  such that  $\|x - \sum_{k=1}^N \alpha_k x_k\| < \varepsilon$  for some scalars  $\alpha_1, \dots, \alpha_N$ .

Consequently, if  $X$  has a Schauder basis, then  $X$  is separable.

*Proof.* The necessary condition is clear.

We are now going to prove the converse statement. Suppose that  $X$  is the closed linear span of a countable subset  $A$ . Now let  $D$  be the linear span of  $A$  over the field  $\mathbb{Q} + i\mathbb{Q}$ . Since  $\mathbb{Q}$  is a countable dense subset of  $\mathbb{R}$ , this implies that  $D$  is a countable dense subset of  $X$ . Thus,  $X$  is separable.

The last statement is clearly follows from the definition of a Schauder basis at once.  $\square$

By Proposition 3.9, we have the following important examples of separable Banach spaces at once.

**Corollary 3.10.** The spaces  $c_0$  and  $\ell^p$  for  $1 \leq p < \infty$  all are separable.

**Remark 3.11.** Proposition 3.9 leads to the following natural question which was first raised by Banach (1932).

**The Basis Problem:** Does every separable Banach space have a Schauder basis?

The answer is "No".

This problem was completely solved by P. Enflo in 1973.

#### 4. LECTURE 4: COMPACT SETS AND FINITE DIMENSIONAL NORMED SPACES

Throughout this section, let  $(x_n)$  be a sequence in a normed space  $X$ . Recall that a subsequence  $(x_{n_k})_{k=1}^{\infty}$  of  $(x_n)$  means that  $(n_k)_{k=1}^{\infty}$  is a sequence of positive integers satisfying  $n_1 < n_2 < \dots < n_k < n_{k+1} < \dots$ , that is, such sequence  $(n_k)$  can be viewed as a strictly increasing function  $\mathbf{n} : k \in \{1, 2, \dots\} \mapsto n_k \in \{1, 2, \dots\}$ .

In this case, note that for each positive integer  $N$ , there is  $K \in \mathbb{N}$  such that  $n_K \geq N$  and thus we have  $n_k \geq N$  for all  $k \geq K$ .

**Definition 4.1.** A subset  $A$  of a normed space  $X$  is said to be compact (more precise, sequentially compact) if every sequence in  $A$  has a convergent subsequence with the limit in  $A$ .

Recall that a subset  $A$  is closed in  $X$  if and only if every convergent sequence  $(x_n)$  in  $A$  implies that  $\lim x_n \in A$ .

**Proposition 4.2.** If  $A$  is a compact subset of  $X$ , then  $A$  is closed and bounded.

*Proof.* It is clear that the result follows if  $A = \emptyset$ . So, we assume that  $A$  is non-empty.

Assume that  $A$  is compact.

We first claim that  $A$  is closed. Let  $(x_n)$  be a sequence in  $A$ . Then by the compactness of  $A$ , there is a convergent subsequence  $(x_{n_k})$  of  $(x_n)$  with  $\lim_k x_{n_k} \in A$ . So, if  $(x_n)$  is convergent, then  $\lim_n x_n = \lim_k x_{n_k} \in A$ . Therefore,  $A$  is closed.

Next, we are going to show the boundedness of  $A$ . Suppose that  $A$  is not bounded. Fix an element  $x_1 \in A$ . Since  $A$  is not bounded, we can find an element  $x_2 \in A$  such that  $\|x_2 - x_1\| > 1$ . Similarly, there is an element  $x_3 \in A$  such that  $\|x_3 - x_k\| > 1$  for  $k = 1, 2$ . To repeat the same step, we can obtain a sequence  $(x_n)$  in  $A$  such that  $\|x_n - x_m\| > 1$  for  $m \neq n$ . From this, we see that the sequence  $(x_n)$  does not have a convergent subsequence. In fact, if  $(x_n)$  has a convergent subsequence  $(x_{n_k})$ . Therefore,  $(x_{n_k})_{k=1}^{\infty}$  is a Cauchy sequence in  $X$ . Then we can find a pair of sufficient large positive integers  $p$  and  $q$  with  $p \neq q$  such that  $\|x_{n_p} - x_{n_q}\| < 1/2$ . It leads to a contradiction because  $\|x_{n_p} - x_{n_q}\| > 1$  by the choice of the sequence  $(x_n)$ . Thus,  $A$  is bounded.  $\square$

The following is an important characterization of a compact set in the the case  $X = \mathbb{R}$ . **Warning:** this result is not true for a general normed space  $X$ .

Let us first recall the following important theorem in real line.

**Theorem 4.3. (Bolzano-Weierstrass Theorem)** Every bounded sequence in  $\mathbb{R}$  has a convergent subsequence.

*Proof.* See [1, Theorem 3.4.8].  $\square$

**Theorem 4.4.** Let  $A$  be a closed subset of  $\mathbb{R}$ . Then the following statements are equivalent.

- (i)  $A$  is compact.
- (ii)  $A$  is closed and bounded.

*Proof.* Part (i)  $\Rightarrow$  (ii) follows from Proposition 4.2 immediately.

It remains to show (ii)  $\Rightarrow$  (i). Suppose that  $A$  is closed and bounded.

Let  $(x_n)$  be a sequence in  $A$ . Thus,  $(x_n)$ . Then the Bolzano-Weierstrass Theorem assures that there is a convergent subsequence  $(x_{n_k})$ . Then by the closeness of  $A$ ,  $\lim_k x_{n_k} \in A$ . Thus  $A$  is compact.

The proof is finished.  $\square$

**Definition 4.5.** We say that two norms  $\|\cdot\|$  and  $\|\cdot\|'$  on a vector space  $X$  are equivalent, write  $\|\cdot\| \sim \|\cdot\|'$ , if there are positive numbers  $c_1$  and  $c_2$  such that  $c_1\|\cdot\| \leq \|\cdot\|' \leq c_2\|\cdot\|$  on  $X$ .

**Example 4.6.** Consider the norms  $\|\cdot\|_1$  and  $\|\cdot\|_\infty$  on  $\ell^1$ . We are going to show that  $\|\cdot\|_1$  and  $\|\cdot\|_\infty$  are not equivalent. In fact, if we put  $x_n(i) := (1, 1/2, \dots, 1/n, 0, 0, \dots)$  for  $n, i = 1, 2, \dots$ . Then  $x_n \in \ell^1$  for all  $n$ . Notice that  $(x_n)$  is a Cauchy sequence with respect to the norm  $\|\cdot\|_\infty$  but it is not a Cauchy sequence with respect to the norm  $\|\cdot\|_1$ . Hence  $\|\cdot\|_1 \not\sim \|\cdot\|_\infty$  on  $\ell^1$ .

**Proposition 4.7.** All norms on a finite dimensional vector space are equivalent.

*Proof.* Let  $X$  be a finite dimensional vector space and let  $\{e_1, \dots, e_N\}$  be a vector base of  $X$ . For each  $x = \sum_{i=1}^N \alpha_i e_i$  for  $\alpha_i \in \mathbb{K}$ , define  $\|x\|_0 = \sum_{i=1}^N |\alpha_i|$ . Then  $\|\cdot\|_0$  is a norm  $X$ . The result is obtained by showing that all norms  $\|\cdot\|$  on  $X$  are equivalent to  $\|\cdot\|_0$ .

Notice that for each  $x = \sum_{i=1}^N \alpha_i e_i \in X$ , we have  $\|x\| \leq (\max_{1 \leq i \leq N} \|e_i\|) \|x\|_0$ . It remains to find  $c > 0$  such that  $c\|\cdot\|_0 \leq \|\cdot\|$ . In fact, let  $\mathbb{K}^N$  be equipped with the sup-norm  $\|\cdot\|_\infty$ , that is  $\|(\alpha_1, \dots, \alpha_N)\|_\infty = \max_{1 \leq i \leq N} |\alpha_i|$ . Define a real-valued function  $f$  on the unit sphere  $S_{\mathbb{K}^N}$  of  $\mathbb{K}^N$  by

$$f : (\alpha_1, \dots, \alpha_N) \in S_{\mathbb{K}^N} \mapsto \|\alpha_1 e_1 + \dots + \alpha_N e_N\|.$$

Notice that the map  $f$  is continuous and  $f > 0$ . It is clear that  $S_{\mathbb{K}^N}$  is compact with respect to the sup-norm  $\|\cdot\|_\infty$  on  $\mathbb{K}^N$ . Hence, there is  $c > 0$  such that  $f(\alpha) \geq c > 0$  for all  $\alpha \in S_{\mathbb{K}^N}$ . This gives  $\|x\| \geq c\|x\|_0$  for all  $x \in X$  as desired. The proof is finished.  $\square$

The following result is clear. The proof is omitted here.

**Lemma 4.8.** Let  $X$  be a normed space. Then the closed unit ball  $B_X$  is compact if and only if every bounded sequence in  $X$  has a convergent subsequence.

**Proposition 4.9.** We have the following assertions.

- (i) All finite dimensional normed spaces are Banach spaces. Consequently, any finite dimensional subspace of a normed space must be closed.
- (ii) The closed unit ball of any finite dimensional normed space is compact.

*Proof.* Let  $(X, \|\cdot\|)$  be a finite dimensional normed space. With the notation as in the proof of Proposition 4.7 above, we see that  $\|\cdot\|$  must be equivalent to the norm  $\|\cdot\|_0$ . It is clear that  $X$  is complete with respect to the norm  $\|\cdot\|_0$  and so is complete in the original norm  $\|\cdot\|$ . The Part (i) follows.

For Part (ii), by using Lemma 4.8, we need to show that any bounded sequence has a convergent subsequence. Let  $(x_n)$  be a bounded sequence in  $X$ . Since all norms on a finite dimensional normed space are equivalent, it suffices to show that  $(x_n)$  has a convergent subsequence with respect to the norm  $\|\cdot\|_0$ .

Using the notation as in Proposition 4.7, for each  $x_n$ , put  $x_n = \sum_{k=1}^N \alpha_{n,k} e_k$ ,  $n = 1, 2, \dots$ . Then by the definition of the norm  $\|\cdot\|_0$ , we see that  $(\alpha_{n,k})_{n=1}^\infty$  is a bounded sequence in  $\mathbb{K}$  for each  $k = 1, 2, \dots, N$ . Then by the Bolzano-Weierstrass Theorem, for each  $k = 1, \dots, N$ , we can find a

convergent subsequence  $(\alpha_{n_j, k})_{j=1}^{\infty}$  of  $(\alpha_{n, k})_{n=1}^{\infty}$ . Put  $\gamma_k := \lim_{j \rightarrow \infty} \alpha_{n_j, k} \in \mathbb{K}$ , for  $k = 1, \dots, N$ . Put  $x := \sum_{k=1}^N \gamma_k e_k$ . Then by the definition of the norm  $\|\cdot\|_0$ , we see that  $\|x_{n_j} - x\|_0 \rightarrow 0$  as  $j \rightarrow \infty$ . Thus,  $(x_n)$  has a convergent subsequence as desired.

The proof is complete.  $\square$

In the rest of this section, we are going to show the converse of Proposition 4.9 (ii) also holds. Before showing the main theorem in this section, we need the following useful result.

**Lemma 4.10. Riesz's Lemma:** *Let  $Y$  be a closed proper subspace of a normed space  $X$ . Then for each  $\theta \in (0, 1)$ , there is an element  $x_0 \in S_X$  such that  $d(x_0, Y) := \inf\{\|x_0 - y\| : y \in Y\} \geq \theta$ .*

*Proof.* Let  $u \in X - Y$  and  $d := \inf\{\|u - y\| : y \in Y\}$ . Notice that since  $Y$  is closed,  $d > 0$  and hence, we have  $0 < d < \frac{d}{\theta}$  because  $0 < \theta < 1$ . This implies that there is  $y_0 \in Y$  such that  $0 < d \leq \|u - y_0\| < \frac{d}{\theta}$ . Now put  $x_0 := \frac{u - y_0}{\|u - y_0\|} \in S_X$ . We are going to show that  $x_0$  is as desired. Indeed, let  $y \in Y$ . Since  $y_0 + \|u - y_0\|y \in Y$ , we have

$$\|x_0 - y\| = \frac{1}{\|u - y_0\|} \|u - (y_0 + \|u - y_0\|y)\| \geq d/\|u - y_0\| > \theta.$$

So,  $d(x_0, Y) \geq \theta$ .  $\square$

**Remark 4.11.** *The Riesz's lemma does not hold when  $\theta = 1$ .*

**Theorem 4.12.** *Let  $X$  be a normed space. Then the following statements are equivalent.*

- (i)  $X$  is a finite dimensional normed space.
- (ii) The closed unit ball  $B_X$  of  $X$  is compact.
- (iii) Every bounded sequence in  $X$  has convergent subsequence.

*Proof.* The implication (i)  $\Rightarrow$  (ii) follows from Proposition 4.9 (ii) at once.

Lemma 4.8 gives the implication (ii)  $\Rightarrow$  (iii).

Finally, for the implication (iii)  $\Rightarrow$  (i), assume that  $X$  is of infinite dimension. Fix an element  $x_1 \in S_X$ . Let  $Y_1 = \mathbb{K}x_1$ . Then  $Y_1$  is a proper closed subspace of  $X$ . The Riesz's lemma gives an element  $x_2 \in S_X$  such that  $\|x_1 - x_2\| \geq 1/2$ . Now consider  $Y_2 = \text{span}\{x_1, x_2\}$ . Then  $Y_2$  is a proper closed subspace of  $X$  since  $\dim X = \infty$ . To apply the Riesz's Lemma again, there is  $x_3 \in S_X$  such that  $\|x_3 - x_k\| \geq 1/2$  for  $k = 1, 2$ . To repeat the same step, there is a sequence  $(x_n) \in S_X$  such that  $\|x_m - x_n\| \geq 1/2$  for all  $n \neq m$ . Thus,  $(x_n)$  is a bounded sequence but it has no convergent subsequence by using the similar argument as in Proposition 4.2. So, the condition (iii) does not hold if  $\dim X = \infty$ . The proof is finished.  $\square$

## 5. LECTURE 5: BOUNDED LINEAR OPERATORS

**Proposition 5.1.** *Let  $T$  be a linear operator from a normed space  $X$  into a normed space  $Y$ . Then the following statements are equivalent.*

- (i)  $T$  is continuous on  $X$ .
- (ii)  $T$  is continuous at  $0 \in X$ .
- (iii)  $\sup\{\|Tx\| : x \in B_X\} < \infty$ .

*In this case, let  $\|T\| = \sup\{\|Tx\| : x \in B_X\}$  and  $T$  is said to be bounded.*

*Proof.* (i)  $\Rightarrow$  (ii) is obvious.

For (ii)  $\Rightarrow$  (i), suppose that  $T$  is continuous at 0. Let  $x_0 \in X$ . Let  $\varepsilon > 0$ . Then there is  $\delta > 0$  such that  $\|Tw\| < \varepsilon$  for all  $w \in X$  with  $\|w\| < \delta$ . Therefore, we have  $\|Tx - Tx_0\| = \|T(x - x_0)\| < \varepsilon$  for

any  $x \in X$  with  $\|x - x_0\| < \delta$ . So, (i) follows.

For (ii)  $\Rightarrow$  (iii), since  $T$  is continuous at 0, there is  $\delta > 0$  such that  $\|Tx\| < 1$  for any  $x \in X$  with  $\|x\| < \delta$ . Now for any  $x \in B_X$  with  $x \neq 0$ , we have  $\|\frac{\delta}{2}x\| < \delta$ . So, we see we have  $\|T(\frac{\delta}{2}x)\| < 1$  and hence, we have  $\|Tx\| < 2/\delta$ . So, (iii) follows.

Finally, it remains to show (iii)  $\Rightarrow$  (ii). Notice that by the assumption of (iii), there is  $M > 0$  such that  $\|Tx\| \leq M$  for all  $x \in B_X$ . So, for each  $x \in X$ , we have  $\|Tx\| \leq M\|x\|$ . This implies that  $T$  is continuous at 0. The proof is complete.  $\square$

**Corollary 5.2.** *Let  $T : X \rightarrow Y$  be a bounded linear map. Then we have*

$$\sup\{\|Tx\| : x \in B_X\} = \sup\{\|Tx\| : x \in S_X\} = \inf\{M > 0 : \|Tx\| \leq M\|x\|, \forall x \in X\}.$$

*Proof.* Let  $a = \sup\{\|Tx\| : x \in B_X\}$ ,  $b = \sup\{\|Tx\| : x \in S_X\}$  and  $c = \inf\{M > 0 : \|Tx\| \leq M\|x\|, \forall x \in X\}$ .

It is clear that  $b \leq a$ . Now for each  $x \in B_X$  with  $x \neq 0$ , then we have  $b \geq \|T(x/\|x\|)\| = (1/\|x\|)\|Tx\| \geq \|Tx\|$ . So, we have  $b \geq a$  and thus,  $a = b$ .

Now if  $M > 0$  satisfies  $\|Tx\| \leq M\|x\|, \forall x \in X$ , then we have  $\|Tw\| \leq M$  for all  $w \in S_X$ . So, we have  $b \leq M$  for all such  $M$ . So, we have  $b \leq c$ . Finally, it remains to show  $c \leq b$ . Notice that by the definition of  $b$ , we have  $\|Tx\| \leq b\|x\|$  for all  $x \in X$ . So,  $c \leq b$ .  $\square$

**Proposition 5.3.** *Let  $X$  and  $Y$  be normed spaces. Let  $B(X, Y)$  be the set of all bounded linear maps from  $X$  into  $Y$ . For each element  $T \in B(X, Y)$ , let*

$$\|T\| = \sup\{\|Tx\| : x \in B_X\}.$$

*be defined as in Proposition 5.1.*

*Then  $(B(X, Y), \|\cdot\|)$  becomes a normed space.*

*Furthermore, if  $Y$  is a Banach space, then so is  $B(X, Y)$ .*

*In particular, if  $Y = \mathbb{K}$ , then  $B(X, \mathbb{K})$  is a Banach space. In this case, put  $X^* := B(X, \mathbb{K})$  and call it the **dual space** of  $X$ .*

*Proof.* One can directly check that  $B(X, Y)$  is a normed space (**Do It By Yourself!**).

We are going to show that  $B(X, Y)$  is complete if  $Y$  is a Banach space. Let  $(T_n)$  be a Cauchy sequence in  $B(X, Y)$ . Then for each  $x \in X$ , it is easy to see that  $(T_n x)$  is also a Cauchy sequence in  $Y$ . So,  $\lim T_n x$  exists in  $Y$  for each  $x \in X$  because  $Y$  is complete. Hence, one can define a map  $Tx := \lim T_n x \in Y$  for each  $x \in X$ . It is clear that  $T$  is a linear map from  $X$  into  $Y$ .

It needs to show that  $T \in B(X, Y)$  and  $\|T - T_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . Let  $\varepsilon > 0$ . Since  $(T_n)$  is a Cauchy sequence in  $B(X, Y)$ , there is a positive integer  $N$  such that  $\|T_m - T_n\| < \varepsilon$  for all  $m, n \geq N$ . So, we have  $\|(T_m - T_n)(x)\| < \varepsilon$  for all  $x \in B_X$  and  $m, n \geq N$ . Taking  $m \rightarrow \infty$ , we have  $\|Tx - T_n x\| \leq \varepsilon$  for all  $n \geq N$  and  $x \in B_X$ . Therefore, we have  $\|T - T_n\| \leq \varepsilon$  for all  $n \geq N$ . From this, we see that  $T - T_N \in B(X, Y)$  and thus,  $T = T_N + (T - T_N) \in B(X, Y)$  and  $\|T - T_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore,  $\lim_n T_n = T$  exists in  $B(X, Y)$ .  $\square$

**Proposition 5.4.** *Let  $X$  and  $Y$  be normed spaces. Suppose that  $X$  is of finite dimension  $n$ . Then we have the following assertions.*

(i) *Any linear operator from  $X$  into  $Y$  must be bounded.*

(ii) *If  $T_k : X \rightarrow Y$  is a sequence of linear operators such that  $T_k x \rightarrow 0$  for all  $x \in X$ , then  $\|T_k\| \rightarrow 0$ .*

*Proof.* Using Proposition 4.7 and the notation as in the proof, then there is  $c > 0$  such that

$$\sum_{i=1}^n |\alpha_i| \leq c \left\| \sum_{i=1}^n \alpha_i e_i \right\|$$

for all scalars  $\alpha_1, \dots, \alpha_n$ . Therefore, for any linear map  $T$  from  $X$  to  $Y$ , we have

$$\|Tx\| \leq \left( \max_{1 \leq i \leq n} \|Te_i\| \right) c \|x\|$$

for all  $x \in X$ . This gives the assertions (i) and (ii) immediately.  $\square$

**Proposition 5.5.** *Let  $Y$  be a closed subspace of  $X$  and  $X/Y$  be the quotient space. For each element  $x \in X$ , put  $\bar{x} := x + Y \in X/Y$  the corresponding element in  $X/Y$ . Define*

$$(5.1) \quad \|\bar{x}\| = \inf\{\|x + y\| : y \in Y\}.$$

*If we let  $\pi : X \rightarrow X/Y$  be the natural projection, that is  $\pi(x) = \bar{x}$  for all  $x \in X$ , then  $(X/Y, \|\cdot\|)$  is a normed space and  $\pi$  is bounded with  $\|\pi\| \leq 1$ . In particular,  $\|\pi\| = 1$  as  $Y$  is a proper closed subspace.*

*Furthermore, if  $X$  is a Banach space, then so is  $X/Y$ .*

*In this case, we call  $\|\cdot\|$  in (5.1) the quotient norm on  $X/Y$ .*

*Proof.* Notice that since  $Y$  is closed, one can directly check that  $\|\bar{x}\| = 0$  if and only if  $x \in Y$ , that is,  $\bar{x} = \bar{0} \in X/Y$ . It is easy to check the other conditions of the definition of a norm. So,  $X/Y$  is a normed space. Also, it is clear that  $\pi$  is bounded with  $\|\pi\| \leq 1$  by the definition of the quotient norm on  $X/Y$ .

Furthermore, if  $Y \subsetneq X$ , then by using the Riesz's Lemma 4.10, we see that  $\|\pi\| = 1$  at once.

We are going to show the last assertion. Suppose that  $X$  is a Banach space. Let  $(\bar{x}_n)$  be a Cauchy sequence in  $X/Y$ . It suffices to show that  $(\bar{x}_n)$  has a convergent subsequence in  $X/Y$  by using Lemma 3.2.

Indeed, since  $(\bar{x}_n)$  is a Cauchy sequence, we can find a subsequence  $(\bar{x}_{n_k})$  of  $(\bar{x}_n)$  such that

$$\|\bar{x}_{n_{k+1}} - \bar{x}_{n_k}\| < 1/2^k$$

for all  $k = 1, 2, \dots$ . Then by the definition of quotient norm, there is an element  $y_1 \in Y$  such that  $\|x_{n_2} - x_{n_1} + y_1\| < 1/2$ . Notice that we have,  $\overline{x_{n_1} - y_1} = \bar{x}_{n_1}$  in  $X/Y$ . So, there is  $y_2 \in Y$  such that  $\|x_{n_2} - y_2 - (x_{n_1} - y_1)\| < 1/2$  by the definition of quotient norm again. Also, we have  $\overline{x_{n_2} - y_2} = \bar{x}_{n_2}$ . Then we also have an element  $y_3 \in Y$  such that  $\|x_{n_3} - y_3 - (x_{n_2} - y_2)\| < 1/2^2$ . To repeat the same step, we can obtain a sequence  $(y_k)$  in  $Y$  such that

$$\|x_{n_{k+1}} - y_{k+1} - (x_{n_k} - y_k)\| < 1/2^k$$

for all  $k = 1, 2, \dots$ . Therefore,  $(x_{n_k} - y_k)$  is a Cauchy sequence in  $X$  and thus,  $\lim_k (x_{n_k} - y_k)$  exists in  $X$  while  $X$  is a Banach space. Set  $x = \lim_k (x_{n_k} - y_k)$ . On the other hand, notice that we have  $\pi(x_{n_k} - y_k) = \pi(x_{n_k})$  for all  $k = 1, 2, \dots$ . This tells us that  $\lim_k \pi(x_{n_k}) = \lim_k \pi(x_{n_k} - y_k) = \pi(x) \in X/Y$  since  $\pi$  is bounded. So,  $(\bar{x}_{n_k})$  is a convergent subsequence of  $(\bar{x}_n)$  in  $X/Y$ . The proof is complete.  $\square$

**Corollary 5.6.** *Let  $T : X \rightarrow Y$  be a linear map. Suppose that  $Y$  is of finite dimension. Then  $T$  is bounded if and only if  $\ker T := \{x \in X : Tx = 0\}$ , the kernel of  $T$ , is closed.*

*Proof.* The necessary part is clear.

Now assume that  $\ker T$  is closed. Then by Proposition 5.5,  $X/\ker T$  becomes a normed space. Also, it is known that there is a linear injection  $\tilde{T} : X/\ker T \rightarrow Y$  such that  $T = \tilde{T} \circ \pi$ , where  $\pi : X \rightarrow X/\ker T$  is the natural projection. Since  $\dim Y < \infty$  and  $\tilde{T}$  is injective,  $\dim X/\ker T < \infty$ . This implies that  $\tilde{T}$  is bounded by Proposition 5.4. Hence  $T$  is bounded because  $T = \tilde{T} \circ \pi$  and  $\pi$  is bounded.  $\square$

**Remark 5.7.** The converse of Corollary 5.6 does not hold when  $Y$  is of infinite dimension. For example, let  $X := \{x \in \ell^2 : \sum_{n=1}^{\infty} n^2 |x(n)|^2 < \infty\}$  (notice that  $X$  is a vector space **Why?**) and  $Y = \ell^2$ . Both  $X$  and  $Y$  are endowed with  $\|\cdot\|_2$ -norm.

Define  $T : X \rightarrow Y$  by  $Tx(n) = nx(n)$  for  $x \in X$  and  $n = 1, 2, \dots$ . Then  $T$  is an unbounded operator (**Check !!**). Notice that  $\ker T = \{0\}$  and hence,  $\ker T$  is closed. So, the closeness of  $\ker T$  does not imply the boundedness of  $T$  in general.

We say that two normed spaces  $X$  and  $Y$  are said to be *isomorphic* (resp. *isometric isomorphic*) if there is a bi-continuous linear isomorphism (resp. isometric) between  $X$  and  $Y$ . We also write  $X = Y$  if  $X$  and  $Y$  are isometric isomorphic.

**Remark 5.8.** Notice that the inverse of a bounded linear isomorphism may not be bounded.

**Example 5.9.** Let  $X := \{f \in C^\infty(-1, 1) : f^{(n)} \in C^b(-1, 1) \text{ for all } n = 0, 1, 2, \dots\}$  and  $Y := \{f \in X : f(0) = 0\}$ . Also,  $X$  and  $Y$  both are equipped with the sup-norm  $\|\cdot\|_\infty$ . Define an operator  $S : X \rightarrow Y$  by

$$Sf(x) := \int_0^x f(t) dt$$

for  $f \in X$  and  $x \in (-1, 1)$ . Then  $S$  is a bounded linear isomorphism but its inverse  $S^{-1}$  is unbounded. In fact, the inverse  $S^{-1} : Y \rightarrow X$  is given by

$$S^{-1}g := g'$$

for  $g \in Y$ .

## 6. LECTURE 6: DUAL SPACES I

All spaces  $X, Y, Z, \dots$  are normed spaces over the field  $\mathbb{K}$  throughout this section. By Proposition 5.3, we have the following assertion at once.

**Proposition 6.1.** Let  $X$  be a normed space. Put  $X^* = B(X, \mathbb{K})$ . Then  $X^*$  is a Banach space and is called the dual space of  $X$ .

**Example 6.2.** Let  $X = \mathbb{K}^N$ . Consider the usual Euclidean norm on  $X$ , that is,  $\|(x_1, \dots, x_N)\| := \sqrt{|x_1|^2 + \dots + |x_N|^2}$ . Define  $\theta : \mathbb{K}^N \rightarrow (\mathbb{K}^N)^*$  by  $\theta x(y) = x_1 y_1 + \dots + x_N y_N$  for  $x = (x_1, \dots, x_N)$  and  $y = (y_1, \dots, y_N) \in \mathbb{K}^N$ . Notice that  $\theta x(y) = \langle x, y \rangle$ , the usual inner product on  $\mathbb{K}^N$ . Then by the Cauchy-Schwarz inequality, it is easy to see that  $\theta$  is an isometric isomorphism. Therefore, we have  $\mathbb{K}^N = (\mathbb{K}^N)^*$ .

**Example 6.3.** Define a map  $T : \ell^1 \rightarrow c_0^*$  by

$$(Tx)(\eta) = \sum_{i=1}^{\infty} x(i)\eta(i)$$

for  $x \in \ell^1$  and  $\eta \in c_0$ .

Then  $T$  is isometric isomorphism and hence,  $c_0^* = \ell^1$ .

*Proof.* The proof is divided into the following steps.

**Step 1.**  $Tx \in c_0^*$  for all  $x \in \ell^1$ .

In fact, let  $\eta \in c_0$ . Then

$$|Tx(\eta)| \leq \left| \sum_{i=1}^{\infty} x(i)\eta(i) \right| \leq \sum_{i=1}^{\infty} |x(i)||\eta(i)| \leq \|x\|_1 \|\eta\|_\infty.$$

So, *Step 1* follows.

**Step 2.**  $T$  is an isometry.

Notice that by *Step 1*, we have  $\|Tx\| \leq \|x\|_1$  for all  $x \in \ell^1$ . It needs to show that  $\|Tx\| \geq \|x\|_1$  for all  $x \in \ell^1$ . Fix  $x \in \ell^1$ . Now for each  $k = 1, 2, \dots$ , consider the polar form  $x(k) = |x(k)|e^{i\theta_k}$ . Notice that  $\eta_n := (e^{-i\theta_1}, \dots, e^{-i\theta_n}, 0, 0, \dots) \in c_0$  for all  $n = 1, 2, \dots$ . Then we have

$$\sum_{k=1}^n |x(k)| = \sum_{k=1}^n x(k)\eta_n(k) = Tx(\eta_n) = |Tx(\eta_n)| \leq \|Tx\|$$

for all  $n = 1, 2, \dots$ . So, we have  $\|x\|_1 \leq \|Tx\|$ .

**Step 3.**  $T$  is a surjection.

Let  $\phi \in c_0^*$  and let  $e_k \in c_0$  be given by  $e_k(j) = 1$  if  $j = k$ , otherwise, is equal to 0. Put  $x(k) := \phi(e_k)$  for  $k = 1, 2, \dots$  and consider the polar form  $x(k) = |x(k)|e^{i\theta_k}$  as above. Then we have

$$\sum_{k=1}^n |x(k)| = \phi\left(\sum_{k=1}^n e^{-i\theta_k} e_k\right) \leq \|\phi\| \left\| \sum_{k=1}^n e^{-i\theta_k} e_k \right\|_\infty = \|\phi\|$$

for all  $n = 1, 2, \dots$ . Therefore,  $x \in \ell^1$ .

Finally, we need to show that  $Tx = \phi$  and thus,  $T$  is surjective. In fact, if  $\eta = \sum_{k=1}^\infty \eta(k)e_k \in c_0$ , then we have

$$\phi(\eta) = \sum_{k=1}^\infty \eta(k)\phi(e_k) = \sum_{k=1}^\infty \eta(k)x_k = Tx(\eta).$$

So, the proof is finished by the *Steps 1-3* above.  $\square$

**Example 6.4.** We have the other important examples of the dual spaces.

(i)  $(\ell^1)^* = \ell^\infty$ .

(ii) For  $1 < p < \infty$ ,  $(\ell^p)^* = \ell^q$ , where  $\frac{1}{p} + \frac{1}{q} = 1$ .

(iii) For a locally compact Hausdorff space  $X$ ,  $C_0(X)^* = M(X)$ , where  $M(X)$  denotes the space of all regular Borel measures on  $X$ .

Parts (i) and (ii) can be obtained by the similar argument as in Example 6.3 (see also in [3, Chapter 8]). Part (iii) is known as the *Riesz representation Theorem* which is referred to [3, Section 21.5] for the details.

In the rest of this section, we are going to show the Hahn-Banach Theorem which is a very important Theorem in mathematics. Before showing this theorem, we need the following lemma first.

**Lemma 6.5.** *Let  $Y$  be a subspace of  $X$  and  $v \in X \setminus Y$ . Let  $Z = Y \oplus \mathbb{K}v$  be the linear span of  $Y$  and  $v$  in  $X$ . If  $f \in Y^*$ , then there is an extension  $F \in Z^*$  of  $f$  such that  $\|F\| = \|f\|$ .*

*Proof.* We may assume that  $\|f\| = 1$  by considering the normalization  $f/\|f\|$  if  $f \neq 0$ .

*Case  $\mathbb{K} = \mathbb{R}$ :*

We first note that since  $\|f\| = 1$ , we have  $|f(x) - f(y)| \leq \|(x+v) - (y+v)\|$  for all  $x, y \in Y$ . This implies that  $-f(x) - \|x+v\| \leq -f(y) + \|y+v\|$  for all  $x, y \in Y$ . Now let  $\gamma = \sup\{-f(x) - \|x+v\| : x \in X\}$ . This implies that  $\gamma$  exists and

$$(6.1) \quad -f(y) - \|y+v\| \leq \gamma \leq -f(y) + \|y+v\|$$

for all  $y \in Y$ . We define  $F : Z \rightarrow \mathbb{R}$  by  $F(y + \alpha v) := f(y) + \alpha\gamma$ . It is clear that  $F|_Y = f$ . For showing  $F \in Z^*$  with  $\|F\| = 1$ , since  $F|_Y = f$  on  $Y$  and  $\|f\| = 1$ , it needs to show  $|F(y + \alpha v)| \leq \|y + \alpha v\|$  for all  $y \in Y$  and  $\alpha \in \mathbb{R}$ .

In fact, for  $y \in Y$  and  $\alpha > 0$ , then by inequality 6.1, we have

$$(6.2) \quad |F(y + \alpha v)| = |f(y) + \alpha\gamma| \leq \|y + \alpha v\|.$$

Since  $y$  and  $\alpha$  are arbitrary in inequality 6.2, we see that  $|F(y + \alpha v)| \leq \|y + \alpha v\|$  for all  $y \in Y$  and  $\alpha \in \mathbb{R}$ . Therefore the result holds when  $\mathbb{K} = \mathbb{R}$ .

Now for the complex case, let  $h = \operatorname{Re} f$  and  $g = \operatorname{Im} f$ . Then  $f = h + ig$  and  $f, g$  both are real linear with  $\|h\| \leq 1$ . Note that since  $f(iy) = if(y)$  for all  $y \in Y$ , we have  $g(y) = -h(iy)$  for all  $y \in Y$ . This gives  $f(\cdot) = h(\cdot) - ih(i\cdot)$  on  $Y$ . Then by the real case above, there is a real linear extension  $H$  on  $Z := Y \oplus \mathbb{R}v \oplus i\mathbb{R}v$  of  $h$  such that  $\|H\| = \|h\|$ . Now define  $F : Z \rightarrow \mathbb{C}$  by  $F(\cdot) := H(\cdot) - iH(i\cdot)$ . Then  $F \in Z^*$  and  $F|_Y = f$ . Thus it remains to show that  $\|F\| = \|f\| = 1$ . It needs to show that  $|F(z)| \leq \|z\|$  for all  $z \in Z$ . Note for  $z \in Z$ , consider the polar form  $F(z) = re^{i\theta}$ . Then  $F(e^{-i\theta}z) = r \in \mathbb{R}$  and thus  $F(e^{-i\theta}z) = H(e^{-i\theta}z)$ . This yields that

$$|F(z)| = r = |F(e^{-i\theta}z)| = |H(e^{-i\theta}z)| \leq \|H\| \|e^{-i\theta}z\| \leq \|z\|.$$

The proof is finished.  $\square$

**Remark 6.6.** Before completing the proof of the Hahn-Banach Theorem, Let us first recall one of super important results in mathematics, called *Zorn's Lemma*, a very humble name. Every mathematics student should know it.

**Zorn's Lemma:** Let  $\mathcal{X}$  be a non-empty set with a partially order " $\leq$ ". Assume that every totally order subset  $\mathcal{C}$  of  $\mathcal{X}$  has an upper bound, i.e. there is an element  $\mathfrak{z} \in \mathcal{X}$  such that  $c \leq \mathfrak{z}$  for all  $c \in \mathcal{C}$ . Then  $\mathcal{X}$  must contain a maximal element  $\mathfrak{m}$ , that is, if  $\mathfrak{m} \leq x$  for some  $x \in \mathcal{X}$ , then  $\mathfrak{m} = x$ .

The following is the typical argument of applying the Zorn's Lemma.

**Theorem 6.7. Hahn-Banach Theorem :** Let  $X$  be a normed space and let  $Y$  be a subspace of  $X$ . If  $f \in Y^*$ , then there exists a linear extension  $F \in X^*$  of  $f$  such that  $\|F\| = \|f\|$ .

*Proof.* Let  $\mathcal{X}$  be the collection of the pairs  $(Y_1, f_1)$ , where  $Y \subseteq Y_1$  is a subspace of  $X$  and  $f_1 \in Y_1^*$  such that  $f_1|_Y = f$  and  $\|f_1\|_{Y_1^*} = \|f\|_{Y^*}$ . Define a partial order  $\leq$  on  $\mathcal{X}$  by  $(Y_1, f_1) \leq (Y_2, f_2)$  if  $Y_1 \subseteq Y_2$  and  $f_2|_{Y_1} = f_1$ . Then by the Zorn's lemma, there is a maximal element  $(\tilde{Y}, F)$  in  $\mathcal{X}$ . The maximality of  $(\tilde{Y}, F)$  and Lemma 6.5 will give  $\tilde{Y} = X$ . The proof is finished.  $\square$

**Proposition 6.8.** Let  $X$  be a normed space and  $x_0 \in X$ . Then there is  $f \in X^*$  with  $\|f\| = 1$  such that  $f(x_0) = \|x_0\|$ . Consequently, we have

$$\|x_0\| = \sup\{|g(x)| : g \in B_{X^*}\}.$$

Also, if  $x, y \in X$  with  $x \neq y$ , then there exists  $f \in X^*$  such that  $f(x) \neq f(y)$ .

*Proof.* Let  $Y = \mathbb{K}x_0$ . Define  $f_0 : Y \rightarrow \mathbb{K}$  by  $f_0(\alpha x_0) := \alpha \|x_0\|$  for  $\alpha \in \mathbb{K}$ . Then  $f_0 \in Y^*$  with  $\|f_0\| = \|x_0\|$ . So, the result follows from the Hahn-Banach Theorem at once.  $\square$

**Remark 6.9.** Proposition 6.8 tells us that the dual space  $X^*$  of  $X$  must be non-zero. Indeed, the dual space  $X^*$  is very "Large" so that it can separate any pair of distinct points in  $X$ .

Furthermore, for any normed space  $Y$  and any pair of points  $x_1, x_2 \in X$  with  $x_1 \neq x_2$ , we can find an element  $T \in B(X, Y)$  such that  $Tx_1 \neq Tx_2$ . In fact, fix a non-zero element  $y \in Y$ . Then by Proposition 6.8, there is  $f \in X^*$  such that  $f(x_1) \neq f(x_2)$ . So, if we define  $Tx = f(x)y$ , then  $T \in B(X, Y)$  as desired.

**Proposition 6.10.** With the notation as above, if  $M$  is closed subspace and  $v \in X \setminus M$ , then there is  $f \in X^*$  such that  $f(M) \equiv 0$  and  $f(v) \neq 0$ .

*Proof.* Since  $M$  is a closed subspace of  $X$ , we can consider the quotient space  $X/M$ . Let  $\pi : X \rightarrow X/M$  be the natural projection. Notice that  $\bar{v} := \pi(v) \neq 0 \in X/M$  because  $v \in X \setminus M$ . Then by Corollary 6.8, there is a non-zero element  $\bar{f} \in (X/M)^*$  such that  $\bar{f}(\bar{v}) \neq 0$ . So, the linear functional  $f := \bar{f} \circ \pi \in X^*$  is as desired.  $\square$

**Proposition 6.11.** *Using the notation as above, if  $X^*$  is separable, then  $X$  is separable.*

*Proof.* Let  $F := \{f_1, f_2, \dots\}$  be a dense subset of  $X^*$ . Then there is a sequence  $(x_n)$  in  $X$  with  $\|x_n\| = 1$  and  $|f_n(x_n)| \geq 1/2\|f_n\|$  for all  $n$ . Now let  $M$  be the closed linear span of  $x_n$ 's. Then  $M$  is a separable closed subspace of  $X$ . We are going to show that  $M = X$ .

Suppose not. Proposition 6.10 will give us a non-zero element  $f \in X^*$  such that  $f(M) \equiv 0$ . From this, we first see that  $f \neq f_m$  for all  $m = 1, 2, \dots$  because  $f(x_m) = 0$  and  $f_m(x_m) \neq 0$  for all  $m = 1, 2, \dots$ . Also, notice that  $B(f, r) \cap F$  must be infinite for all  $r > 0$ . So, there is a subsequence  $(f_{n_k})$  such that  $\|f_{n_k} - f\| \rightarrow 0$ . This gives

$$\frac{1}{2}\|f_{n_k}\| \leq |f_{n_k}(x_{n_k})| = |f_{n_k}(x_{n_k}) - f(x_{n_k})| \leq \|f_{n_k} - f\| \rightarrow 0$$

because  $f(M) \equiv 0$ . So  $\|f_{n_k}\| \rightarrow 0$  and hence  $f = 0$ . It leads to a contradiction again. Thus, we can conclude that  $M = X$  as desired.  $\square$

**Remark 6.12.** The converse of Proposition 6.11 does not hold. For example, consider  $X = \ell^1$ . Then  $\ell^1$  is separable but the dual space  $(\ell^1)^* = \ell^\infty$  is not.

## 7. LECTURE 7: DUAL SPACES II

All notation are as in Section 6

**Proposition 7.1.** *Let  $X$  and  $Y$  be normed spaces. For each element  $T \in B(X, Y)$ , define a linear operator  $T^* : Y^* \rightarrow X^*$  by*

$$T^*y^*(x) := y^*(Tx)$$

for  $y^* \in Y^*$  and  $x \in X$ . Then  $T^* \in B(Y^*, X^*)$  and  $\|T^*\| = \|T\|$ . In this case,  $T^*$  is called the adjoint operator of  $T$ .

*Proof.* We first claim that  $\|T^*\| \leq \|T\|$  and hence,  $\|T^*\|$  is bounded.

In fact, for any  $y^* \in Y^*$  and  $x \in X$ , we have  $|T^*y^*(x)| = |y^*(Tx)| \leq \|y^*\|\|Tx\|$ . So,  $\|T^*y^*\| \leq \|T\|\|y^*\|$  for all  $y^* \in Y^*$ . Thus,  $\|T^*\| \leq \|T\|$ .

It remains to show  $\|T\| \leq \|T^*\|$ . Let  $x \in B_X$ . Then by Proposition 6.8, there is  $y^* \in S_{Y^*}$  such that  $\|Tx\| = |y^*(Tx)| = |T^*y^*(x)| \leq \|T^*y^*\| \leq \|T^*\|$ . This implies that  $\|T\| \leq \|T^*\|$ .  $\square$

**Example 7.2.** Let  $X$  and  $Y$  be the finite dimensional normed spaces. Let  $(e_i)_{i=1}^n$  and  $(f_j)_{j=1}^m$  be the bases for  $X$  and  $Y$  respectively. Let  $\theta_X : X \rightarrow X^*$  and  $\theta_Y : Y \rightarrow Y^*$  be the identifications as in Example 6.2. Let  $e_i^* := \theta_X e_i \in X^*$  and  $f_j^* := \theta_Y f_j \in Y^*$ . Then  $e_i^*(e_l) = \delta_{il}$  and  $f_j^*(f_l) = \delta_{jl}$ , where,  $\delta_{il} = 1$  if  $i = l$ ; otherwise is 0.

Now if  $T \in B(X, Y)$  and  $(a_{ij})_{m \times n}$  is the representative matrix of  $T$  corresponding to the bases  $(e_i)_{i=1}^n$  and  $(f_j)_{j=1}^m$  respectively, then  $a_{kl} = f_k^*(Te_l) = T^*f_k^*(e_l)$ . Therefore, if  $(a'_{lk})_{n \times m}$  is the representative matrix of  $T^*$  corresponding to the bases  $(f_j^*)$  and  $(e_i^*)$ , then  $a_{kl} = a'_{lk}$ . Hence the transpose  $(a_{kl})^t$  is the the representative matrix of  $T^*$ .

**Proposition 7.3.** *For a normed space  $X$ , let  $Q : X \rightarrow X^{**}$  be the canonical map, that is,  $Qx(x^*) := x^*(x)$  for  $x^* \in X^*$  and  $x \in X$ . Then  $Q$  is an isometry.*

*Proof.* Note that for  $x \in X$  and  $x^* \in B_{X^*}$ , we have  $|Q(x)(x^*)| = |x^*(x)| \leq \|x\|$ . Then  $\|Q(x)\| \leq \|x\|$ .

It remains to show that  $\|x\| \leq \|Q(x)\|$  for all  $x \in X$ . In fact, for  $x \in X$ , there is  $x^* \in X^*$  with  $\|x^*\| = 1$  such that  $\|x\| = |x^*(x)| = |Q(x)(x^*)|$  by Proposition 6.8. Thus we have  $\|x\| \leq \|Q(x)\|$ . The proof is finished.  $\square$

**Remark 7.4.** Let  $T : X \rightarrow Y$  be a bounded linear operator and  $T^{**} : X^{**} \rightarrow Y^{**}$  the second dual operator induced by the adjoint operator of  $T$ . With notation as in Proposition 7.3 above, the following diagram always commutes.

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ Q_X \downarrow & & \downarrow Q_Y \\ X^{**} & \xrightarrow{T^{**}} & Y^{**} \end{array}$$

**Definition 7.5.** A normed space  $X$  is said to be reflexive if the canonical map  $Q : X \rightarrow X^{**}$  is surjective. (Notice that every reflexive space must be a Banach space.)

**Example 7.6.** We have the following examples.

- (i) : Every finite dimensional normed space  $X$  is reflexive.
- (ii) :  $\ell^p$  is reflexive for  $1 < p < \infty$ .
- (iii) :  $c_0$  and  $\ell^1$  are not reflexive.

*Proof.* For Part (i), if  $\dim X < \infty$ , then  $\dim X = \dim X^{**}$ . Hence, the canonical map  $Q : X \rightarrow X^{**}$  must be surjective.

Part (ii) follows from  $(\ell^p)^* = \ell^q$  for  $1 < p < \infty$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ .

For Part (iii), notice that  $c_0^{**} = (\ell^1)^* = \ell^\infty$ . Since  $\ell^\infty$  is non-separable but  $c_0$  is separable. So, the canonical map  $Q$  from  $c_0$  to  $c_0^{**} = \ell^\infty$  must not be surjective.

For the case of  $\ell^1$ , we have  $(\ell^1)^{**} = (\ell^\infty)^*$ . Since  $\ell^\infty$  is non-separable, the dual space  $(\ell^\infty)^*$  is non-separable by Proposition 6.11. So,  $\ell^1 \neq (\ell^1)^{**}$ .  $\square$

**Proposition 7.7.** Every closed subspace of a reflexive space is reflexive.

*Proof.* Let  $Y$  be a closed subspace of a reflexive space  $X$ . Let  $Q_Y : Y \rightarrow Y^{**}$  and  $Q_X : X \rightarrow X^{**}$  be the canonical maps as before. Let  $y_0^{**} \in Y^{**}$ . We define an element  $\phi \in X^{**}$  by  $\phi(x^*) := y_0^{**}(x^*|_Y)$  for  $x^* \in X^*$ . Since  $X$  is reflexive, there is  $x_0 \in X$  such that  $Q_X x_0 = \phi$ . Suppose  $x_0 \notin Y$ . Then by Proposition 6.10, there is  $x_0^* \in X^*$  such that  $x_0^*(x_0) \neq 0$  but  $x_0^*(Y) \equiv 0$ . Note that we have  $x_0^*(x_0) = Q_X x_0(x_0^*) = \phi(x_0^*) = y_0^{**}(x_0^*|_Y) = 0$ . It leads to a contradiction. So,  $x_0 \in Y$ . The proof is finished if we have  $Q_Y(x_0) = y_0^{**}$ .

In fact, for each  $y^* \in Y^*$ , then by the Hahn-Banach Theorem,  $y^*$  has a continuous extension  $x^*$  in  $X^*$ . Then we have

$$Q_Y(x_0)(y^*) = y^*(x_0) = x^*(x_0) = Q_X(x_0)(x^*) = \phi(x^*) = y_0^{**}(x^*|_Y) = y_0^{**}(y^*).$$

$\square$

**Example 7.8.** By using Proposition 7.7, we immediately see that the space  $\ell^\infty$  is not reflexive because it contains a non-reflexive closed subspace  $c_0$ .

## 8. LECTURE 8: OPEN MAPPING THEOREM

Throughout this section, let  $X$  and  $Y$  be normed spaces.

Recall that a subset  $V$  of  $X$  is said to be open if for each element  $x \in V$ , there is  $r > 0$  such that  $B(x, r) \subseteq V$ .

**Definition 8.1.** A linear map  $T : X \rightarrow Y$  is called an open map if  $T(V)$  is an open subset of  $Y$  whenever  $V$  is an open subset of  $X$ .

The following theorem is one of important theorems in Functional Analysis.

**Theorem 8.2. Open Mapping Theorem** Suppose that  $X$  and  $Y$  both are Banach spaces. If  $T$  is a bounded linear surjection from  $X$  onto  $Y$ , then  $T$  is an open map.

**Lemma 8.3.** Let  $T : X \rightarrow Y$  be a bounded linear isomorphism. Then the inverse  $T^{-1} : Y \rightarrow X$  is bounded if and only if  $T$  is an open map.

*Proof.* We first assume that the inverse  $T^{-1}$  is bounded. Let  $V$  be an open subset of  $X$ . We claim that  $T(V)$  is an open subset of  $Y$ . Let  $b \in T(V)$  and  $a = T^{-1}(b) \in V$ . Since  $V$  is open, there is  $r > 0$  such that  $B(a, r) \subseteq V$ . On the other hand, since the map  $T^{-1}$  is continuous at  $b$ , there is  $\delta > 0$  such that  $\|T^{-1}(y) - T^{-1}(b)\| < r$  whenever  $\|y - b\| < \delta$ . Therefore, if we let  $y = Tx$  and  $y \in B(b, \delta)$ , then  $x \in B(a, r) \subseteq V$  and thus,  $B(b, \delta) \subseteq T(V)$ . Therefore,  $T(V)$  is open.

For the converse, assume that  $T$  is an open map. It suffices to show that the inverse map  $T^{-1}$  is continuous at 0. Let  $\varepsilon > 0$ . Then by the assumption  $0 \in T(B_X(0, \varepsilon))$  is an open subset of  $Y$  and hence, there is  $\delta > 0$  such that  $B_Y(0, \delta) \subseteq T(B_X(0, \varepsilon))$ . This implies that if  $\|y\| < \delta$ , then  $\|T^{-1}(y)\| < \varepsilon$ . So,  $T^{-1}$  is continuous at 0 as desired.  $\square$

**Remark 8.4.** Example 5.9, together with Lemma 8.3, show that the assumption of the completeness of  $X$  and  $Y$  in the Open Mapping Theorem is essential.

**Corollary 8.5.** Let  $X$  and  $Y$  be Banach spaces. If  $T : X \rightarrow Y$  is a bounded linear isomorphism, then the inverse  $T^{-1} : Y \rightarrow X$  is also bounded.

*Proof.* The assertion follows from the Open Mapping Theorem and Lemma 8.3 at once.  $\square$

**Corollary 8.6.** Let  $\|\cdot\|_1$  and  $\|\cdot\|_2$  be the complete norms on a vector space  $E$ . Suppose that there is  $c > 0$  such that  $\|\cdot\|_2 \leq c\|\cdot\|_1$  on  $E$ . Then  $\|\cdot\|_1 \sim \|\cdot\|_2$ .

*Proof.* Notice that the identity map  $I : (E, \|\cdot\|_1) \rightarrow (E, \|\cdot\|_2)$  is a bounded isomorphism by the assumption. Then the result follows from Corollary 8.5 immediately.  $\square$

## 9. LECTURE 9: CLOSED GRAPH THEOREM

In this section, we are going to show one of the important theorems in functional analysis.

Let  $T : X \rightarrow Y$  be a linear map from a normed space  $X$  into a normed space  $Y$ . The graph of  $T$ , write  $G(T)$ , is defined by the following

$$G(T) := \{(x, Tx) : x \in X\} (\subseteq X \times Y).$$

**Definition 9.1.** With the notation as above, an operator  $T : X \rightarrow Y$  is said to be closed if the graph  $G(T)$  of  $T$  is closed in the following sense:

if  $(x_n)$  is a convergent sequence in  $X$  with  $\lim_n x_n = x \in X$  such that  $\lim Tx_n = y \in Y$  exists, then  $Tx = y$ .

The following result is clear.

**Proposition 9.2.** *Every bounded linear operator must be closed.*

**Remark 9.3.** *The following example shows that the converse of 9.2 does not hold.*

**Example 9.4.** *Let  $X := \{f : (-1, 1) \rightarrow \mathbb{R} : f^{(n)} \text{ exists and bounded for all } n = 0, 1, \dots\}$ .  $X$  is equipped with the sup-norm  $\|\cdot\|_\infty$ . Define  $T : X \rightarrow X$  by  $Tf = f'$ . Then  $T$  is closed but it is not bounded.*

**Theorem 9.5. Closed Graph Theorem** *Let  $X$  and  $Y$  be Banach spaces. Let  $T : X \rightarrow Y$  be a linear operator. Then  $T$  is bounded if and only if  $T$  is closed.*

*Proof.* The necessary condition is clear. For showing the sufficient condition, now we assume that  $T$  is closed. We first define a norm  $\|\cdot\|_0$  on  $X$  by

$$\|x\|_0 := \|x\| + \|Tx\|$$

for  $x \in X$ .

**Claim 1:**  $X$  is complete in the norm  $\|\cdot\|_0$ .

In fact, it is clear that if  $(x_n)$  is a Cauchy sequence in  $X$  with respect to the new norm  $\|\cdot\|_0$ , then so are the sequences  $(x_n)$  and  $(Tx_n)$  with respect to the original norm in  $X$  and  $Y$  respectively. Since  $X$  and  $Y$  both are Banach spaces, we see that  $\lim_n x_n = x$  (in the original norm of  $\|\cdot\|$  on  $X$ ) and  $\lim Tx_n = y$  both exist in  $X$  and  $Y$  respectively. From this we see that  $Tx = y$  by the closeness of  $T$ . Thus, we have

$$\|x_n - x\|_0 = \|x_n - x\| + \|Tx_n - Tx\| = \|x_n - x\| + \|Tx_n - y\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore  $\|\cdot\|_0$  is a complete norm on  $X$ . The Claim 1 follows.

On the other hand, we have  $\|\cdot\| \leq \|\cdot\|_0$  on  $X$ . Then by Corollary 8.6 and Claim 1, we see that  $\|\cdot\| \sim \|\cdot\|_0$  on  $X$  and thus, there is  $c > 0$  such that  $\|\cdot\|_0 \leq c\|\cdot\|$  on  $X$ . Therefore, we have  $\|Tx\| \leq \|x\|_0 \leq c\|x\|$  for all  $x \in X$ . Hence,  $T$  is bounded.  $\square$

**Proposition 9.6.** *Let  $E$  and  $F$  be the closed subspaces of a Banach space  $X$  such that  $X = E \oplus F$ . Define an operator  $P : X \rightarrow X$  by  $Px = u$  if  $x = u + v$  for  $u \in E$  and  $v \in F$  (in this case,  $P$  is called the projection along the decomposition  $X = E \oplus F$ ). Then  $P$  is bounded.*

*Proof.* Suppose that  $(x_n)$  is a convergent sequence in  $X$  with the limit  $x \in X$  such that  $\lim Px_n = y \in X$ . Put  $x_n = u_n + v_n$  and  $x = u + v$  for  $u_n, u \in E$  and  $v_n, v \in F$ . Since  $u_n = Px_n \rightarrow y$  and  $E$  is closed, we have  $y \in E$ . This implies that  $v_n = x_n - u_n \rightarrow x - y$ . From this we have  $x - y \in F$  because  $v_n \in F$  and  $F$  is closed. This implies that  $Px = y$ . The Closed Graph Theorem will imply that  $P$  is bounded as desired.  $\square$

## 10. LECTURE 10: GEOMETRY OF HILBERT SPACE I

From now on, all vector spaces are over the complex field. Recall that an *inner product* on a vector space  $V$  is a function  $(\cdot, \cdot) : V \times V \rightarrow \mathbb{C}$  which satisfies the following conditions.

- (i)  $(x, x) \geq 0$  for all  $x \in V$  and  $(x, x) = 0$  if and only if  $x = 0$ .
- (ii)  $(x, y) = \overline{(y, x)}$  for all  $x, y \in V$ .
- (iii)  $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$  for all  $x, y, z \in V$  and  $\alpha, \beta \in \mathbb{C}$ .

Consequently, for each  $x \in V$ , the map  $y \in V \mapsto (x, y) \in \mathbb{C}$  is conjugate linear by the conditions (ii) and (iii), that is  $(x, \alpha y + \beta z) = \bar{\alpha}(x, y) + \bar{\beta}(x, z)$  for all  $y, z \in V$  and  $\alpha, \beta \in \mathbb{C}$ . Also, the inner product  $(\cdot, \cdot)$  will give a norm on  $V$  which is defined by

$$\|x\| := \sqrt{(x, x)}$$

for  $x \in V$ .

We first recall the following useful properties of an inner product space which can be found in the standard text books of linear algebras.

**Proposition 10.1.** *Let  $V$  be an inner product space. For all  $x, y \in V$ , we always have:*

(i): **(Cauchy-Schwarz inequality):**  $|(x, y)| \leq \|x\|\|y\|$  Consequently, the inner product on  $V \times V$  is jointly continuous.

(ii): **(Parallelogram law):**  $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$

Furthermore, a norm  $\|\cdot\|$  on a vector space  $X$  is induced by an inner product if and only if it satisfies the Parallelogram law. In this case such inner product is given by the following:

$$\operatorname{Re}(x, y) = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2) \quad \text{and} \quad \operatorname{Im}(x, y) = \frac{1}{4}(\|x + iy\|^2 - \|x - iy\|^2)$$

for all  $x, y \in X$ .

**Example 10.2.** *It follows from Proposition 10.1 immediately that  $\ell^2$  is an inner product space and  $\ell^p$  is not for all  $p \in [1, \infty] \setminus \{2\}$ .*

From now on, all vector spaces are assumed to be a complex inner product spaces. Recall that two vectors  $x$  and  $y$  in an inner product space  $V$  are said to be *orthogonal* if  $(x, y) = 0$ . Also, a set of elements  $\{v_i\}_{i \in I}$  in  $V$  is said to be *orthonormal* if  $(x_i, x_i) = 1$  and  $(x_i, x_j) = 0$  for  $i, j \in I$  with  $i \neq j$ . The following is known in a standard course of linear algebra.

**Proposition 10.3. Gram-Schmidt process** *Let  $\{x_1, x_2, \dots\}$  be a sequence of linearly independent vectors in an inner product space  $V$ . Put  $e_1 := x_1/\|x_1\|$ . Define  $e_n$  inductively on  $n$  by*

$$e_{n+1} := \frac{x_n - \sum_{k=1}^n (x, e_k)e_k}{\|x_n - \sum_{k=1}^n (x, e_k)e_k\|}.$$

*Then  $\{e_n : n = 1, 2, \dots\}$  forms an orthonormal system in  $V$ . Moreover, the linear span of  $x_1, \dots, x_n$  is equal to the linear span of  $e_1, \dots, e_n$  for all  $n = 1, 2, \dots$*

**Proposition 10.4. (Bessel's inequality)** *Let  $\{e_1, \dots, e_N\}$  be an orthonormal set in an inner product space  $V$ , that is  $(e_i, e_j) = 1$  if  $i = j$ , otherwise is equal to 0. Then for any  $x \in V$ , we have*

$$\sum_{i=1}^N |(x, e_i)|^2 \leq \|x\|^2.$$

*Proof.* It can be obtained by the following equality immediately

$$\|x - \sum_{i=1}^N (x, e_i)e_i\|^2 = \|x\|^2 - \sum_{i=1}^N |(x, e_i)|^2.$$

□

**Corollary 10.5.** Let  $(e_i)_{i \in I}$  be an orthonormal set in an inner product space  $V$ . Then for any element  $x \in V$ , the set

$$\{i \in I : (e_i, x) \neq 0\}$$

is countable.

*Proof.* Note that for each  $x \in V$ , we have

$$\{i \in I : (e_i, x) \neq 0\} = \bigcup_{n=1}^{\infty} \{i \in I : |(e_i, x)| \geq 1/n\}.$$

Then the Bessel's inequality implies that the set  $\{i \in I : |(e_i, x)| \geq 1/n\}$  must be finite for each  $n \geq 1$ . So the result follows.  $\square$

The following is one of the most important classes in mathematics.

**Definition 10.6.** A Hilbert space is a Banach space whose norm is given by an inner product.

In the rest of this section,  $X$  always denotes a complex Hilbert space with an inner product  $(\cdot, \cdot)$ .

**Proposition 10.7.** Let  $(e_n)$  be a sequence of orthonormal vectors in a Hilbert space  $X$ . Then for any  $x \in V$ , the series  $\sum_{n=1}^{\infty} (x, e_n)e_n$  is convergent.

Moreover, if  $(e_{\sigma(n)})$  is a rearrangement of  $(e_n)$ , that is,  $\sigma : \{1, 2, \dots\} \rightarrow \{1, 2, \dots\}$  is a bijection. Then we have

$$\sum_{n=1}^{\infty} (x, e_n)e_n = \sum_{n=1}^{\infty} (x, e_{\sigma(n)})e_{\sigma(n)}.$$

*Proof.* Since  $X$  is a Hilbert space, the convergence of the series  $\sum_{n=1}^{\infty} (x, e_n)e_n$  follows from the Bessel's inequality at once. In fact, if we put  $s_p := \sum_{n=1}^p (x, e_n)e_n$ , then we have

$$\|s_{p+k} - s_p\|^2 = \sum_{p+1 \leq n \leq p+k} |(x, e_n)|^2.$$

Now put  $y = \sum_{n=1}^{\infty} (x, e_n)e_n$  and  $z = \sum_{n=1}^{\infty} (x, e_{\sigma(n)})e_{\sigma(n)}$ . Notice that we have

$$\begin{aligned} (y, y - z) &= \lim_N \left( \sum_{n=1}^N (x, e_n)e_n, \sum_{n=1}^N (x, e_n)e_n - z \right) \\ &= \lim_N \sum_{n=1}^N |(x, e_n)|^2 - \lim_N \sum_{n=1}^N (x, e_n) \sum_{j=1}^{\infty} \overline{(x, e_{\sigma(j)})} (e_n, e_{\sigma(j)}) \\ &= \sum_{n=1}^{\infty} |(x, e_n)|^2 - \lim_N \sum_{n=1}^N (x, e_n) \overline{(x, e_n)} \quad (\text{N.B: for each } n, \text{ there is a unique } j \text{ such that } n = \sigma(j)) \\ &= 0. \end{aligned}$$

Similarly, we have  $(z, y - z) = 0$ . The result follows.  $\square$

A family of an orthonormal vectors, say  $\mathcal{B}$ , in  $X$  is said to be **complete** if it is maximal with respect to the set inclusion order, that is, if  $\mathcal{C}$  is another family of orthonormal vectors with  $\mathcal{B} \subseteq \mathcal{C}$ , then  $\mathcal{B} = \mathcal{C}$ .

A complete orthonormal subset of  $X$  is also called an **orthonormal base** of  $X$ .

**Proposition 10.8.** *Let  $\{e_i\}_{i \in I}$  be a family of orthonormal vectors in  $X$ . Then the followings are equivalent:*

- (i):  $\{e_i\}_{i \in I}$  is complete;
- (ii): if  $(x, e_i) = 0$  for all  $i \in I$ , then  $x = 0$ ;
- (iii): for any  $x \in X$ , we have  $x = \sum_{i \in I} (x, e_i) e_i$ ;
- (iv): for any  $x \in X$ , we have  $\|x\|^2 = \sum_{i \in I} |(x, e_i)|^2$ .

In this case, the expression of each element  $x \in X$  in Part (iii) is unique.

**Note :** there are only countable many  $(x, e_i) \neq 0$  by Corollary 10.5, so the sums in (iii) and (iv) are convergent by Proposition 10.7.

**Proposition 10.9.** *Let  $X$  be a Hilbert space. Then*

- (i) :  $X$  possesses an orthonormal base.
- (ii) : If  $\{e_i\}_{i \in I}$  and  $\{f_j\}_{j \in J}$  both are the orthonormal bases for  $X$ , then  $I$  and  $J$  have the same cardinality, that is, there is a bijection from  $I$  onto  $J$ . In this case, the cardinality  $|I|$  of  $I$  is called the orthonormal dimension of  $X$ .

*Proof.* Part (i) follows from Zorn's Lemma at once.

For part (ii), if the cardinality  $|I|$  is finite, then the assertion is clear since  $|I| = \dim X$  (vector space dimension) in this case.

Now assume that  $|I|$  is infinite, for each  $e_i$ , put  $J_{e_i} := \{j \in J : (e_i, f_j) \neq 0\}$ . Note that since  $\{e_i\}_{i \in I}$  is maximal, Proposition 10.8 implies that we have

$$\{f_j\}_{j \in J} \subseteq \bigcup_{i \in I} J_{e_i}.$$

Notice that  $J_{e_i}$  is countable for each  $e_i$  by using Proposition 10.5. On the other hand, we have  $|\mathbb{N}| \leq |I|$  because  $|I|$  is infinite and thus  $|\mathbb{N} \times I| = |I|$ . Then we have

$$|J| \leq \sum_{i \in I} |J_{e_i}| = \sum_{i \in I} |\mathbb{N}| = |\mathbb{N} \times I| = |I|.$$

From symmetry argument, we also have  $|I| \leq |J|$ . □

**Remark 10.10.** *Recall that a vector space dimension of  $X$  is defined by the cardinality of a maximal linearly independent set in  $X$ .*

*Notice that if  $X$  is finite dimensional, then the orthonormal dimension is the same as the vector space dimension.*

*Also, the vector space dimension is larger than the orthonormal dimension in general since every orthogonal set must be linearly independent.*

**Example 10.11.** The followings are important classes of Hilbert spaces.

- (i)  $\mathbb{C}^n$  is a  $n$ -dimensional Hilbert space. In this case, the inner product is given by  $(z, w) := \sum_{k=1}^n z_k \bar{w}_k$  for  $z = (z_1, \dots, z_n)$  and  $(w_1, \dots, w_n)$  in  $\mathbb{C}^n$ .  
The natural basis  $\{e_1, \dots, e_n\}$  forms an orthonormal basis for  $\mathbb{C}^n$ .
- (ii)  $\ell^2$  is a separable Hilbert space of infinite dimension whose inner product is given by  $(x, y) := \sum_{n=1}^{\infty} x(n) \overline{y(n)}$  for  $x, y \in \ell^2$ .  
If we put  $e_n(n) = 1$  and  $e_n(k) = 0$  for  $k \neq n$ , then  $\{e_n\}$  is an orthonormal basis for  $\ell^2$ .
- (iii) Let  $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ . For each  $f \in C(\mathbb{T})$  (the space of all complex-valued continuous functions defined on  $\mathbb{T}$ ), the integral of  $f$  is defined by

$$\int_{\mathbb{T}} f(z) dz := \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) dt = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} f(e^{it}) dt + \frac{i}{2\pi} \int_0^{2\pi} \operatorname{Im} f(e^{it}) dt.$$

An inner product on  $C(\mathbb{T})$  is given by

$$(f, g) := \int_{\mathbb{T}} f(z) \overline{g(z)} dz$$

for each  $f, g \in C(\mathbb{T})$ . We write  $\|\cdot\|_2$  for the norm induced by this inner product.

The Hilbert space  $L^2(\mathbb{T})$  is defined by the completion of  $C(\mathbb{T})$  under the norm  $\|\cdot\|_2$ .

Now for each  $n \in \mathbb{Z}$ , put  $f_n(z) = z^n$ . We claim that  $\{f_n : n = 0, \pm 1, \pm 2, \dots\}$  is an orthonormal basis for  $L^2(\mathbb{T})$ .

In fact, by using the Euler Formula:  $e^{i\theta} = \cos \theta + i \sin \theta$  for  $\theta \in \mathbb{R}$ , we see that the family  $\{f_n : n \in \mathbb{Z}\}$  is orthonormal.

It remains to show that the family  $\{f_n\}$  is maximal. By Proposition 10.9, it needs to show that if  $(g, f_n) = 0$  for all  $n \in \mathbb{Z}$ , then  $g = 0$  in  $L^2(\mathbb{T})$ . For showing this, we have to make use of the known fact that every element in  $L^2(\mathbb{T})$  can be approximated by the polynomial functions on  $\mathbb{Z}$  in  $\|\cdot\|_2$ -norm due to the *Stone-Weierstrass Theorem*. Thus, we can find a sequence of polynomials  $(p_n)$  such that  $\|g - p_n\|_2 \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $(g, f_n) = 0$  for all  $n$ , we see that  $(g, p_n) = 0$  for all  $n$ . Therefore, we have

$$\|g\|_2^2 = \lim_n (g, p_n) = 0.$$

The proof is finished.

We say that two Hilbert spaces  $X$  and  $Y$  are said to be *isomorphic* if there is linear isomorphism  $U$  from  $X$  onto  $Y$  such that  $(Ux, Ux') = (x, x')$  for all  $x, x' \in X$ . In this case  $U$  is called a *unitary operator*.

**Theorem 10.12.** *Two Hilbert spaces are isomorphic if and only if they have the same orthonormal dimension.*

*Proof.* The converse part ( $\Leftarrow$ ) is clear.

Now for the ( $\Rightarrow$ ) part, let  $X$  and  $Y$  be isomorphic Hilbert spaces. Let  $U : X \rightarrow Y$  be a unitary. Note that if  $\{e_i\}_{i \in I}$  is an orthonormal base of  $X$ , then  $\{Ue_i\}_{i \in I}$  is also an orthonormal base of  $Y$ . Thus the necessary part follows from Proposition 10.9 at once.  $\square$

**Corollary 10.13.** *The Hilbert spaces  $L^2(\mathbb{T})$  and  $\ell^2$  are isomorphic.*

*Proof.* In Examples 10.11 (ii) and (iii), we have shown that the Hilbert spaces  $L^2(\mathbb{T})$  and  $\ell^2$  have the same orthonormal dimension. Then by Theorem 10.14 above, we see that  $L^2(\mathbb{T})$  and  $\ell^2$  are isomorphic.  $\square$

**Corollary 10.14.** *Every separable Hilbert space is isomorphic to  $\ell^2$  or  $\mathbb{C}^n$  for some  $n$ .*

*Proof.* Let  $X$  be a separable Hilbert space.

If  $\dim X < \infty$ , then it is clear that  $X$  is isomorphic to  $\mathbb{C}^n$  for  $n = \dim X$ .

Now suppose that  $\dim X = \infty$  and its orthonormal dimension is larger than  $|\mathbb{N}|$ , that is  $X$  has an orthonormal base  $\{f_i\}_{i \in I}$  with  $|I| > |\mathbb{N}|$ . Note that since  $\|f_i - f_j\| = \sqrt{2}$  for all  $i, j \in I$  with  $i \neq j$ . This implies that  $B(e_i, 1/4) \cap B(e_j, 1/4) = \emptyset$  for  $i \neq j$ .

On the other hand, if we let  $D$  be a countable dense subset of  $X$ , then  $B(f_i, 1/4) \cap D \neq \emptyset$  for all  $i \in I$ . So for each  $i \in I$ , we can pick up an element  $x_i \in D \cap B(f_i, 1/4)$ . Therefore, one can define an injection from  $I$  into  $D$ . It is absurd to the countability of  $D$ .  $\square$

## 11. LECTURE 11: GEOMETRY OF HILBERT SPACE II

In this section, let  $X$  always denote a complex Hilbert space. Recall that a subset  $D$  of  $X$  is said to be convex if it satisfies the condition:  $tx + (1-t)y \in D$  whenever  $x, y \in D$  and  $t \in [0, 1]$ . In particular, every subspace or ball in  $X$  is convex.

**Proposition 11.1.** *If  $D$  is a closed convex subset of  $X$ , then there is a unique element  $z \in D$  such that*

$$\|z\| = \inf\{\|x\| : x \in D\}.$$

*Consequently, for any element  $u \in X$ , there is a unique element  $w \in D$  such that*

$$\|u - w\| = d(u, D) := \inf\{\|u - x\| : x \in D\}.$$

*Proof.* We first claim the existence of such  $z$ .

Let  $d := \inf\{\|x\| : x \in D\}$ . Then there is a sequence  $(x_n)$  in  $D$  such that  $\|x_n\| \rightarrow d$ . Notice that  $(x_n)$  is a Cauchy sequence. In fact, the Parallelogram Law implies that

$$\left\|\frac{x_m - x_n}{2}\right\|^2 = \frac{1}{2}\|x_m\|^2 + \frac{1}{2}\|x_n\|^2 - \left\|\frac{x_m + x_n}{2}\right\|^2 \leq \frac{1}{2}\|x_m\|^2 + \frac{1}{2}\|x_n\|^2 - d^2 \rightarrow 0$$

as  $m, n \rightarrow \infty$ , where the last inequality holds because  $D$  is convex and hence  $\frac{1}{2}(x_m + x_n) \in D$ . Let  $z := \lim_n x_n$ . Then  $\|z\| = d$  and  $z \in D$  because  $D$  is closed.

For the uniqueness, let  $z, z' \in D$  such that  $\|z\| = \|z'\| = d$ . Thanks to the Parallelogram Law again, we have

$$\left\|\frac{z - z'}{2}\right\|^2 = \frac{1}{2}\|z\|^2 + \frac{1}{2}\|z'\|^2 - \left\|\frac{z + z'}{2}\right\|^2 \leq \frac{1}{2}\|z\|^2 + \frac{1}{2}\|z'\|^2 - d^2 = 0.$$

Therefore  $z = z'$ .

The last assertion follows by considering the closed convex set  $u - D := \{u - x : x \in D\}$  immediately.  $\square$

**Proposition 11.2.** *Suppose that  $M$  is a closed subspace. Let  $u \in X$  and  $w \in M$ . Then the followings are equivalent:*

- (i):  $\|u - w\| = d(u, M)$ ;
- (ii):  $u - w \perp M$ , that is  $(u - w, x) = 0$  for all  $x \in M$ .

*Consequently, for each element  $u \in X$ , there is a unique element  $w \in M$  such that  $u - w \perp M$ .*

*Proof.* Let  $d := d(u, M)$ .

For proving (i)  $\Rightarrow$  (ii), fix an element  $x \in M$ . Then for any  $t > 0$ , note that since  $w + tx \in M$ , we have

$$d^2 \leq \|u - w - tx\|^2 = \|u - w\|^2 + \|tx\|^2 - 2\operatorname{Re}(u - w, tx) = d^2 + \|tx\|^2 - 2\operatorname{Re}(u - w, tx).$$

This implies that

$$(11.1) \quad 2\operatorname{Re}(u - w, x) \leq t\|x\|^2$$

for all  $t > 0$  and for all  $x \in M$ . So by considering  $-x$  in Eq.11.1, we obtain

$$2|\operatorname{Re}(u - w, x)| \leq t\|x\|^2.$$

for all  $t > 0$ . This implies that  $\operatorname{Re}(u - w, x) = 0$  for all  $x \in M$ . Similarly, putting  $\pm ix$  into Eq.11.1, we have  $\operatorname{Im}(u - w, x) = 0$ . So (ii) follows.

For (ii)  $\Rightarrow$  (i), we need to show that  $\|u - w\|^2 \leq \|u - x\|^2$  for all  $x \in M$ . Note that since  $u - w \perp M$  and  $w \in M$ , we have  $u - w \perp w - x$  for all  $x \in M$ . This gives

$$\|u - x\|^2 = \|(u - w) + (w - x)\|^2 = \|u - w\|^2 + \|w - x\|^2 \geq \|u - w\|^2.$$

Part (i) follows.

The last statement is obtained by Proposition 11.1 immediately.  $\square$

Now for each subspace  $M$  of  $X$ , the orthogonal set of  $M$ , write  $M^\perp$ , is defined by

$$M^\perp := \{x \in X : x \perp M\}.$$

It is clear that  $M^\perp$  is a closed subspace of  $X$ . Moreover, we have the following important property of a Hilbert space.

**Theorem 11.3.** *Let  $M$  be a closed subspace. Then we have  $X = M \oplus M^\perp$ . In this case,  $M^\perp$  is called the orthogonal complement of  $M$ .*

*Proof.* It is clear that  $M \cap M^\perp = (0)$ . It remains to show  $X = M + M^\perp$ .

Let  $u \in X$ . Then by Proposition 11.2, we can find an element  $w \in M$  such that  $u - w \perp M$ . Thus  $u - w \in M^\perp$  and  $u = w + (u - w)$ . The proof is finished.  $\square$

**Corollary 11.4.** *With the notation as above, an element  $x_0 \notin M$  if and only if there is an element  $m \in M$  such that  $x_0 - m \perp M$ .*

*Proof.* It is clear from Theorem 11.3.  $\square$

**Corollary 11.5.** *If  $M$  is a closed subspace of  $X$ , then  $M^{\perp\perp} = M$ .*

*Proof.* It is clear that  $M \subseteq M^{\perp\perp}$  by the definition of  $M^{\perp\perp}$ . Now if there is  $x \in M^{\perp\perp} \setminus M$ , then by the decomposition  $X = M \oplus M^\perp$  obtained in Theorem 11.3, we have  $x = y + z$  for some  $y \in M$  and  $z \in M^\perp$ . This implies that  $z = x - y \in M^\perp \cap M^{\perp\perp} = (0)$ . This gives  $x = y \in M$ . It leads to a contradiction.  $\square$

**Remark 11.6.** It is worthwhile pointing out that for a general Banach space  $X$  and a closed subspace  $M$  of  $X$ , it **May Not** have a complementary **Closed** subspace  $N$  in  $X$ , that is  $X = M \oplus N$ , for example,  $c_0$  does not have a closed complementary subspace in  $\ell^\infty$  (not obvious fact). If  $M$  has a complementary closed subspace  $N$ , we say that  $M$  is complemented in  $X$ .

**Example 11.7.** Let  $L^2(\mathbb{T})$  be the Hilbert space defined as in Example 10.11. Define a linear functional  $\varphi : L^2(\mathbb{T}) \rightarrow \mathbb{C}$  by

$$\varphi(f) := \int_{\mathbb{T}} f(z) dz$$

for  $f \in L^2(\mathbb{T})$ . Put  $M = \ker \varphi$ . Notice that  $\varphi$  is bounded and thus,  $M$  is a closed subspace. Also, note that  $M = \{f \in L^2(\mathbb{T}) : \int_{\mathbb{T}} f(z) dz = 0\}$ .

Let  $g \in L^2(\mathbb{T})$ . Then by Proposition 11.2, there is a unique element  $h \in M$  such that

$$\|g - h\| = d(g, M).$$

Next, we are going to find the element  $h \in M$ .

In fact, by using Proposition 11.2 again, we have  $g - h \perp M$  and thus,  $g - h \in M^\perp$ . On the other hand, we see that  $M = (\mathbb{C}\mathbf{1})^\perp$  where  $\mathbf{1}(z) \equiv 1$  for all  $z \in \mathbb{T}$ . From this, we see that  $M^\perp = (\mathbb{C}\mathbf{1})^{\perp\perp} = \mathbb{C}\mathbf{1}$  by Corollary 11.5. Therefore, we have  $g - h = \lambda\mathbf{1}$  for some  $\lambda \in \mathbb{C}$ . It remains to determine such  $\lambda$ . Since  $h \in M$ , we see that

$$\int_{\mathbb{T}} g(z) dz = \int_{\mathbb{T}} (g - h)(z) dz = \lambda \int_{\mathbb{T}} \mathbf{1}(z) dz = \lambda.$$

We can now conclude that if  $g \in L^2(\mathbb{T})$ , then the element

$$h := g - \left( \int_{\mathbb{T}} g(z) dz \right) \mathbf{1} \in M$$

satisfies  $\|g - h\| = d(g, M)$  as desired.

## 12. LECTURE 12: RIESZ REPRESENTATION THEOREM

Throughout this section, let  $X$  be a Hilbert space.

**Theorem 12.1. Riesz Representation Theorem :** *For each  $f \in X^*$ , then there is a unique element  $v_f \in X$  such that*

$$f(x) = (x, v_f)$$

for all  $x \in X$  and we have  $\|f\| = \|v_f\|$ .

Furthermore, if  $(e_i)_{i \in I}$  is an orthonormal base of  $X$ , then  $v_f = \sum_i \overline{f(e_i)} e_i$ .

*Proof.* We first prove the uniqueness of  $v_f$ . If  $z \in X$  also satisfies the condition:  $f(x) = (x, z)$  for all  $x \in X$ . This implies that  $(x, z - v_f) = 0$  for all  $x \in X$ . So  $z - v_f = 0$ .

Now for proving the existence of  $v_f$ , it suffices to show the case  $\|f\| = 1$ . Then  $\ker f$  is a closed proper subspace. Then by the orthogonal decomposition again, we have

$$X = \ker f \oplus (\ker f)^\perp.$$

Since  $f \neq 0$ , we have  $(\ker f)^\perp$  is linear isomorphic to  $\mathbb{C}$ . Also note that the restriction of  $f$  on  $(\ker f)^\perp$  is of norm one. Hence there is an element  $v_f \in (\ker f)^\perp$  with  $\|v_f\| = 1$  such that  $f(v_f) = \|f|_{(\ker f)^\perp}\| = 1$  and  $(\ker f)^\perp = \mathbb{C}v_f$ . So for each element  $x \in X$ , we have  $x = z + \alpha v_f$  for some  $z \in \ker f$  and  $\alpha \in \mathbb{C}$ . Then  $f(x) = \alpha f(v_f) = \alpha = (x, v_f)$  for all  $x \in X$ .

Concerning about the last assertion, if we put  $v_f = \sum_{i \in I} \alpha_i e_i$ , then  $f(e_j) = (e_j, v_f) = \overline{\alpha_j}$  for all  $j \in I$ . The proof is finished.  $\square$

**Corollary 12.2.** *With the notation as in Theorem 12.1, Define the map*

$$(12.1) \quad \Phi : f \in X^* \mapsto v_f \in X, \text{ i.e., } f(y) = (y, \Phi(f))$$

for all  $y \in X$  and  $f \in X^*$ .

And if we define  $(f, g)_{X^*} := (v_g, v_f)_X$  for  $f, g \in X^*$ . Then  $(X^*, (\cdot, \cdot)_{X^*})$  becomes a Hilbert space. Moreover,  $\Phi$  is an anti-unitary operator from  $X^*$  onto  $X$ , that is  $\Phi$  satisfies the conditions:

$$\Phi(\alpha f + \beta g) = \overline{\alpha} \Phi(f) + \overline{\beta} \Phi(g) \quad \text{and} \quad (\Phi f, \Phi g)_X = (g, f)_{X^*}$$

for all  $f, g \in X^*$  and  $\alpha, \beta \in \mathbb{C}$ .

Furthermore, if we define  $J : x \in X \mapsto f_x \in X^*$ , where  $f_x(y) := (y, x)$ , then  $J$  is the inverse of  $\Phi$ , and hence,  $J$  is an isometric conjugate linear isomorphism.

*Proof.* The result follows immediately from the observation that  $v_{f+g} = v_f + v_g$  and  $v_{\alpha f} = \overline{\alpha} v_f$  for all  $f \in X^*$  and  $\alpha \in \mathbb{C}$ .

The last assertion is clearly obtained by the Eq.12.1 above.  $\square$

**Corollary 12.3.** *Every Hilbert space is reflexive.*

*Proof.* Using the notation as in the Riesz Representation Theorem 12.1, let  $X$  be a Hilbert space. and  $Q : X \rightarrow X^{**}$  the canonical isometry. Let  $\psi \in X^{**}$ . To apply the Riesz Theorem on the dual space  $X^*$ , there exists an element  $x_0^* \in X^*$  such that

$$\psi(f) = (f, x_0^*)_{X^*}$$

for all  $f \in X^*$ . By using Corollary 12.2, there is an element  $x_0 \in X$  such that  $x_0 = v_{x_0^*}$  and thus, we have

$$\psi(f) = (f, x_0^*)_{X^*} = (x_0, v_f)_X = f(x_0)$$

for all  $f \in X^*$ . Therefore,  $\psi = Q(x_0)$  and so,  $X$  is reflexive.

The proof is finished.  $\square$

## 13. LECTURE 13: OPERATORS ON A HILBERT SPACE

Throughout this section, all spaces are complex Hilbert spaces. Let  $B(X, Y)$  denote the space of all bounded linear operators from  $X$  into  $Y$ . If  $X = Y$ , write  $B(X)$  for  $B(X, X)$ .

Let  $T \in B(X, Y)$ . We will make use the following simple observation:

$$(13.1) \quad (Tx, y) = 0 \text{ for all } x \in X; y \in Y \quad \text{if and only if} \quad T = 0.$$

Therefore, the elements in  $B(X, Y)$  are uniquely determined by the Eq.13.1, that is,  $T = S$  in  $B(X, Y)$  if and only if  $(Tx, y) = (Sx, y)$  for all  $x \in X$  and  $y \in Y$ .

**Remark 13.1.** For Hilbert spaces  $H_1$  and  $H_2$ , we consider their direct sum  $H := H_1 \oplus H_2$ . If we define the inner product on  $H$  by

$$(x_1 \oplus x_2, y_1 \oplus y_2) := (x_1, y_1)_{H_1} + (x_2, y_2)_{H_2}$$

for  $x_1 \oplus x_2$  and  $y_1 \oplus y_2$  in  $H$ , then  $H$  becomes a Hilbert space. Now for each  $T \in B(H_1, H_2)$ , we can define an element  $\tilde{T} \in B(H)$  by  $\tilde{T}(x_1 \oplus x_2) := 0 \oplus Tx_1$ . So, the space  $B(H_1, H_2)$  can be viewed as a closed subspace of  $B(H)$ . Thus, we can consider the case of  $H_1 = H_2$  for studying the space  $B(H_1, H_2)$ .

**Proposition 13.2.** Let  $T \in B(X)$ . Then we have

(i):  $T = 0$  if and only if  $(Tx, x) = 0$  for all  $x \in X$ . Consequently, for  $T, S \in B(X)$ ,  $T = S$  if and only if  $(Tx, x) = (Sx, x)$  for all  $x \in X$ .

(ii):  $\|T\| = \sup\{|(Tx, y)| : x, y \in X \text{ with } \|x\| = \|y\| = 1\}$ .

*Proof.* It is clear that the necessary part in Part (i). Now we are going to the sufficient part in Part (i), that is we assume that  $(Tx, x) = 0$  for all  $x \in X$ . This implies that we have

$$0 = (T(x + iy), x + iy) = (Tx, x) + i(Ty, x) - i(Tx, y) + (Tiy, iy) = i(Ty, x) - i(Tx, y).$$

So we have  $(Ty, x) - (Tx, y) = 0$  for all  $x, y \in X$ . In particular, if we replace  $y$  by  $iy$  in the equation, then we get  $i(Ty, x) - \bar{i}(Tx, y) = 0$  and hence we have  $(Ty, x) + (Tx, y) = 0$ . Therefore we have  $(Tx, y) = 0$ .

For part (ii) : Let  $\alpha = \sup\{|(Tx, y)| : x, y \in X \text{ with } \|x\| = \|y\| = 1\}$ . It is clear that we have  $\|T\| \geq \alpha$ . It needs to show  $\|T\| \leq \alpha$ .

In fact, for each  $x \in X$  with  $\|x\| = 1$ , if  $Tx \neq 0$ , then we take  $y = Tx/\|Tx\|$ . From this, we see that  $\|Tx\| = |(Tx, y)| \leq \alpha$ . This implies that  $\|T\| \leq \alpha$ . The proof is finished.  $\square$

**Proposition 13.3.** Let  $T \in B(X)$ . Then there is a unique element  $T^*$  in  $B(X)$  such that

$$(13.2) \quad (Tx, y) = (x, T^*y)$$

In this case,  $T^*$  is called the adjoint operator of  $T$ .

*Proof.* We first show the uniqueness. Suppose that there are  $S_1, S_2$  in  $B(X)$  which satisfy the Eq.13.2. Then  $(x, S_1y) = (x, S_2y)$  for all  $x, y \in X$ . Eq.13.1 implies that  $S_1 = S_2$ .

Finally, we prove the existence. Note that if we fix an element  $y \in X$ , define the map  $f_y(x) := (Tx, y)$  for all  $x \in X$ . Then  $f_y \in X^*$ . The Riesz Representation Theorem implies that there is a unique element  $y^* \in X$  such that  $(Tx, y) = (x, y^*)$  for all  $x \in X$  and  $\|f_y\| = \|y^*\|$ . On the other hand, we have

$$|f_y(x)| = |(Tx, y)| \leq \|T\|\|x\|\|y\|$$

for all  $x, y \in X$  and thus  $\|f_y\| \leq \|T\|\|y\|$ . If we put  $T^*(y) := y^*$ , then  $T^*$  satisfies the Eq.13.2. Also, we have  $\|T^*y\| = \|y^*\| = \|f_y\| \leq \|T\|\|y\|$  for all  $y \in X$ . So  $T^* \in B(X)$  with  $\|T^*\| \leq \|T\|$  indeed. Hence  $T^*$  is as desired.  $\square$

**Remark 13.4.** Let  $S, T : X \rightarrow X$  be linear operators (without assuming to be bounded). If they satisfy the Eq.13.2 above, i.e.,

$$(Tx, y) = (x, Sy)$$

for all  $x, y \in X$ . Using the Closed Graph Theorem, one can show that  $S$  and  $T$  both are automatically bounded.

In fact, let  $(x_n)$  be a sequence in  $X$  such that  $\lim x_n = x$  and  $\lim Sx_n = y$  for some  $x, y \in X$ . Now for any  $z \in X$ , we have

$$(z, Sx) = (Tz, x) = \lim(Tz, x_n) = \lim(z, Sx_n) = (z, y).$$

Thus  $Sx = y$  and hence  $S$  is bounded by the Closed Graph Theorem.

Similarly, we can also see that  $T$  is bounded.

**Proposition 13.5.** Let  $T, S \in B(X)$ . Then we have

(i):  $T^* \in B(X)$  and  $\|T^*\| = \|T\|$ .

(ii): The map  $T \in B(X) \mapsto T^* \in B(X)$  is an isometric conjugate anti-isomorphism, that is,

$$(\alpha T + \beta S)^* = \bar{\alpha}T^* + \bar{\beta}S^* \quad \text{for all } \alpha, \beta \in \mathbb{C}; \quad \text{and} \quad (TS)^* = S^*T^*.$$

(iii):  $\|T^*T\| = \|T\|^2$ .

*Proof.* For Part (i), in the proof of Proposition 13.3, we have shown that  $\|T^*\| \leq \|T\|$ . And the reverse inequality clearly follows from  $T^{**} = T$ .

The Part (ii) follows from the adjoint operators are uniquely determined by the Eq.13.2 above.

For Part (iii), we always have  $\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$ . For the reverse inequality, let  $x \in B_X$ . Then

$$\|Tx\|^2 = (Tx, Tx) = (T^*Tx, x) \leq \|T^*Tx\|\|x\| \leq \|T^*T\|.$$

Therefore, we have  $\|T\|^2 \leq \|T^*T\|$ . □

**Example 13.6.** If  $X = \mathbb{C}^n$  and  $D = (a_{ij})_{n \times n}$  an  $n \times n$  matrix, then  $D^* = (\bar{a}_{ji})_{n \times n}$ . In fact, notice that

$$a_{ji} = (De_i, e_j) = (e_i, D^*e_j) = \overline{(D^*e_j, e_i)}.$$

So if we put  $D^* = (d_{ij})_{n \times n}$ , then  $d_{ij} = (D^*e_j, e_i) = \bar{a}_{ji}$ .

**Example 13.7.** Let  $\ell^2(\mathbb{N}) := \{x : \mathbb{N} \rightarrow \mathbb{C} : \sum_{i=0}^{\infty} |x(i)|^2 < \infty\}$ . And put  $(x, y) := \sum_{i=0}^{\infty} x(i)\overline{y(i)}$ .

Define the operator  $D \in B(\ell^2(\mathbb{N}))$  (called the unilateral shift) by

$$Dx(i) = x(i-1)$$

for  $i \in \mathbb{N}$  and where we set  $x(-1) := 0$ , that is  $D(x(0), x(1), \dots) = (0, x(0), x(1), \dots)$ .

Then  $D$  is an isometry and the adjoint operator  $D^*$  is given by

$$D^*x(i) := x(i+1)$$

for  $i = 0, 1, \dots$ , that is  $D^*(x(0), x(1), \dots) = (x(1), x(2), \dots)$ .

Indeed one can directly check that

$$(Dx, y) = \sum_{i=0}^{\infty} x(i-1)\overline{y(i)} = \sum_{j=0}^{\infty} x(j)\overline{y(j+1)} = (x, D^*y).$$

Note that  $D^*$  is NOT an isometry.

**Example 13.8.** Let  $\ell^\infty(\mathbb{N}) = \{x : \mathbb{N} \rightarrow \mathbb{C} : \sup_{i \geq 0} |x(i)| < \infty\}$  and  $\|x\|_\infty := \sup_{i \geq 0} |x(i)|$ . For each  $x \in \ell^\infty$ , define  $M_x \in B(\ell^2(\mathbb{N}))$  by

$$M_x(\xi) := x \cdot \xi$$

for  $\xi \in \ell^2(\mathbb{N})$ , where  $(x \cdot \xi)(i) := x(i)\xi(i); i \in \mathbb{N}$ .

Then  $\|M_x\| = \|x\|_\infty$  and  $M_x^* = M_{\bar{x}}$ , where  $\bar{x}(i) := \overline{x(i)}$ .

**Definition 13.9.** Let  $T \in B(X)$  and let  $I$  be the identity operator on  $X$ .  $T$  is said to be

- (i) : selfadjoint if  $T^* = T$ ;
- (ii) : normal if  $T^*T = TT^*$ ;
- (iii) : unitary if  $T^*T = TT^* = I$ .

**Proposition 13.10.** We have

(i) : Let  $T : X \rightarrow X$  be a linear operator.  $T$  is selfadjoint if and only if

$$(13.3) \quad (Tx, y) = (x, Ty) \quad \text{for all } x, y \in X.$$

(ii) :  $T$  is normal if and only if  $\|Tx\| = \|T^*x\|$  for all  $x \in X$ .

*Proof.* The necessary part of Part (i) is clear.

Now suppose that the Eq.13.3 holds, it needs to show that  $T$  is bounded. Indeed, it follows from Remark13.4 at once.

For Part (ii), note that by Proposition 13.2,  $T$  is normal if and only if  $(T^*Tx, x) = (TT^*x, x)$ . So, Part (ii) follows from that

$$\|Tx\|^2 = (Tx, Tx) = (T^*Tx, x) = (TT^*x, x) = (T^*x, T^*x) = \|T^*x\|^2$$

for all  $x \in X$ . □

**Proposition 13.11.** Let  $T \in B(H)$ . We have the following assertions.

- (i) :  $T$  is selfadjoint if and only if  $(Tx, x) \in \mathbb{R}$  for all  $x \in H$ .
- (ii) : If  $T$  is selfadjoint, then  $\|T\| = \sup\{|(Tx, x)| : x \in H \text{ with } \|x\| = 1\}$ .

*Proof.* Part (i) is clearly follows from Proposition13.2.

For Part (ii), if we let  $a = \sup\{|(Tx, x)| : x \in H \text{ with } \|x\| = 1\}$ , then it is clear that  $a \leq \|T\|$ . We are now going to show the reverse inequality. Since  $T$  is selfadjoint, one can directly check that

$$(T(x+y), x+y) - (T(x-y), x-y) = 4\text{Re}(Tx, y)$$

for all  $x, y \in H$ . Thus if  $x, y \in H$  with  $\|x\| = \|y\| = 1$  and  $(Tx, y) \in \mathbb{R}$ , then by using the Parallelogram Law, we have

$$(13.4) \quad |(Tx, y)| \leq \frac{a}{4}(\|x+y\|^2 + \|x-y\|^2) = \frac{a}{2}(\|x\|^2 + \|y\|^2) = a.$$

Now for  $x, y \in H$  with  $\|x\| = \|y\| = 1$ , by considering the polar form of  $(Tx, y) = re^{i\theta}$ , the Eq.13.4 gives

$$|(Tx, y)| = |(Tx, e^{i\theta}y)| \leq a.$$

Since  $\|T\| = \sup_{\|x\|=\|y\|=1} |(Tx, y)|$ , we have  $\|T\| \leq a$  as desired. The proof is finished. □

**Proposition 13.12.** Let  $T \in B(X)$ . Then we have

$$\ker T = (\text{im}T^*)^\perp \quad \text{and} \quad (\ker T)^\perp = \overline{\text{im}T^*}$$

where  $\text{im}T$  denotes the image of  $T$ .

*Proof.* The first equality clearly follows from  $x \in \ker T$  if and only if  $0 = (Tx, z) = (x, T^*z)$  for all  $z \in X$ .

On the other hand, it is clear that we have  $M^\perp = \overline{M}^\perp$  for any subspace  $M$  of  $X$ . This together with the first equality and Corollary 11.5 will yield the second equality at once.  $\square$

**Proposition 13.13.** *Let  $X$  be a Hilbert space. Let  $M$  and  $N$  be the closed subspaces of  $X$  such that*

$$X = M \oplus N \quad \dots\dots\dots (*)$$

*Let  $Q : X \rightarrow X$  be the projection along the decomposition  $(*)$  with  $\text{im } Q = M$ . Then  $N = M^\perp$  (and hence  $(*)$  is the orthogonal decomposition of  $X$  with respect to  $M$ ) if and only if  $Q$  satisfies the conditions:  $Q^2 = Q$  and  $Q^* = Q$ . And  $Q$  is called the orthogonal projection (or projection for simply) with respect to  $M$ .*

*Proof.* Now if  $N = M^\perp$ , then for  $y, y' \in M$  and  $z, z' \in N$ , we have

$$(Q(y+z), y'+z') = (y, y') = (y+z, Q(y'+z')).$$

So  $Q^* = Q$ .

The converse of the last statement follows from Proposition 13.12 at once because  $\ker Q = N$  and  $\text{im } Q = M$ .

The proof is complete.  $\square$

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